

# TELEPORTATION OF CONTINUOUS QUANTUM VARIABLES: A NEW APPROACH

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ABSTRACT. Teleportation of optical field states (as continuous quantum variables) is usually described in terms of Wigner functions. This is in marked contrast to the theoretical treatment of teleportation of qubits. In this paper we show that by using the holomorphic representation of the canonical commutation relations, teleportation of continuous quantum variables can be treated in complete analogy to the case of teleportation of qubits. In order to emphasize this analogy, short descriptions of the basic experimental schemes both for teleportation of qubits and of continuous variables are included. We conclude our paper with a brief discussion of the effectiveness of our description of continuous variable teleportation and of the role of localization of quantum states in teleportation problems.

## 1. INTRODUCTION

The essence of the quantum information processing consists in (1) generating quantum entanglements among quantum systems, (2) controlling the quantum entanglements. Quantum teleportation contains the essence of quantum information processing ((1) and (2)). Quantum teleportation will play an important role in the realization of the quantum computer.

Quantum teleportation is first discovered in 1993 by Bennett et al. [2]. A few years later two experimental reports on quantum teleportation were published, in 1997 by Bouwmeester et al. [4], and in 1998 by Furusawa et al. [10].

The experiment (by Bouwmeester et al. [4]) for the teleportation of photon states (qubits) is difficult, because the efficiency of single photon experiments is presently restricted in principle due to the inability to identify all four Bell states, and also in practice by the low efficiency of single photon production and detection. In contrast, the important feature of the technique used in the experiment for the teleportation of optical field states by Furusawa et al. [10] is its high efficiency. This is due to the in principle ability to perform the required joint measurements and the technical maturity of optical field detection. The experiment of Furusawa et al. [10] is often considered to be the first experimental realization of quantum teleportation.

The purpose of this lecture is to present teleportation of photon states and optical field states in a unified way, with emphasis on a new approach to teleportation of optical field states (quantum teleportation of continuous quantum variables), based on the holomorphic representation of the canonical commutation relations instead on the use of Wigner functions. Here we try to have our lecture self contained as far as possible (not with respect to the underlying literature but respect to the arguments which are used).

The mathematics for the teleportation of photon states (qubits) is the theory of  $8 \times 8$  matrices, that is, the theory of 8 dimensional Hilbert spaces. However, in order to describe the optical field states we need the theory of infinite dimensional Hilbert spaces. The experiment uses the entangled states of squeezed laser beams with squeezing parameter  $r$ . Since we can neither generate an infinitely squeezed ( $r = \infty$ ) EPR state nor prepare ideal detectors with efficiency 1, we cannot have complete teleportation  $\psi_{\text{out}} = \psi_{\text{in}}$ . We have to measure the quality of the output state  $\psi_{\text{out}}$ . To do so, one often uses the notion of fidelity which is defined by  $F(\rho, \sigma) = \text{tr} \sqrt{\rho^{1/2} \sigma \rho^{1/2}}$  for density operators  $\rho$  and  $\sigma$ .

In the description of teleportation of continuous quantum variables, one often uses the Wigner function which allows a compact form for the output state  $\psi_{\text{out}}$  for a given coherent input state  $\psi_{\text{in}}$  and has an intimate connection with the fidelity (see [6]). Such a formulation sets the theoretical description teleportation of continuous quantum variables apart from the description of teleportation of qubits which is based on the use of state vectors and projective measurements.

In order to understand the essence of the teleportation of continuous quantum variables and to show the close analogy to teleportation of qubits, we use the “holomorphic representation” of the canonical commutation relations (CCR), which was introduced by Bargmann [1]. This representation allows to derive easily explicit formulae for the theoretical description of the basic operations and objects used in the experiment.

The contents of this lecture can briefly be described as follows: In Section 2, we recall quantum teleportation of qubits, in Section 3, we present briefly the experiment of teleportation of qubits. In Section 4, we introduce the holomorphic representation of CCR and determine explicitly the kernels of the mathematical operations (Bogoliubov transformation) which are needed in the Section 5 to derive the mathematical realization of the devices for the manipulation of photon states (squeezed vacuum state, half-beam splitter, displacement operator).

The theory of laser and parametric oscillator (amplifier) are important for quantum teleportation but not included in this lecture. We refer for example to [13, 7]. In Section 6, we present our approach to quantum teleportation of continuous variables. In Section 7, the experiment made by Furusawa et al. [10] is briefly presented with a

discussion of a controversy about this experiment. Some remarks on the notion of locality in the theory of quantum teleportation conclude this lecture.

For the convenience of the reader, the first part of an appendix explains the notion of generalized states (which play an essential rôle in our approach) and mentions some basic properties; the second part gives the detailed proofs for the results presented in Section 4.

## 2. TELEPORTATION OF QUBITS

For the teleportation of qubits

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \Leftrightarrow \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \in \mathbb{C}^2, \quad |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

the following single qubit gates are used:

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad X \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \beta \\ \alpha \end{pmatrix}, \quad X(|0\rangle, |1\rangle) = (|1\rangle, |0\rangle),$$

$$Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = (X+Z)/\sqrt{2},$$

$$H(|0\rangle, |1\rangle) = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle, |0\rangle - |1\rangle).$$

The matrix  $H$  is called the Hadamard gate.

In order to treat multi-qubits, we must consider composite systems. **Axiom of composite system:** The state space of the composite physical system is the tensor product of the state spaces of the component physical systems. Moreover, if we have systems numbered 1 through  $n$ , and system number  $i$  is prepared in the state  $|\psi_i\rangle$ , then the state of the total system is  $|\psi_1\rangle \otimes |\psi_2\rangle \otimes \cdots \otimes |\psi_n\rangle$ .

The two qubit gate (Controlled-Not gate)  $M_{\text{CNOT}}$  will play an important role in our lecture. This gate acts on the basis vectors

$$|ij\rangle = |i\rangle \otimes |j\rangle = |i\rangle|j\rangle$$

as follows:

$$M_{\text{CNOT}}(|00\rangle, |01\rangle, |10\rangle, |11\rangle) = (|00\rangle, |01\rangle, |11\rangle, |10\rangle).$$

**2.1. EPR pair, Bell states.** The essence of quantum information science is quantum entanglement and its manipulation. The entangled states  $|\beta_{ij}\rangle$  called EPR pair or Bell states are created by using

Hadamard gate  $H$  and controlled-not gate  $M_{\text{CNOT}}$ :

(2.1)

$$\begin{aligned} M_{\text{CNOT}}(H \otimes I)|00\rangle &= \frac{1}{\sqrt{2}}M_{\text{CNOT}}(|0\rangle + |1\rangle) \otimes |0\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = |\beta_{00}\rangle, \\ M_{\text{CNOT}}(H \otimes I)|01\rangle &= \frac{1}{\sqrt{2}}M_{\text{CNOT}}(|0\rangle + |1\rangle) \otimes |1\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) = |\beta_{01}\rangle, \\ M_{\text{CNOT}}(H \otimes I)|10\rangle &= \frac{1}{\sqrt{2}}M_{\text{CNOT}}(|0\rangle - |1\rangle) \otimes |0\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) = |\beta_{10}\rangle, \\ M_{\text{CNOT}}(H \otimes I)|11\rangle &= \frac{1}{\sqrt{2}}M_{\text{CNOT}}(|0\rangle - |1\rangle) \otimes |1\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) = |\beta_{11}\rangle. \end{aligned}$$

*Remark 2.1.* The canonical basis  $|ij\rangle$  can be written in terms of the Bell basis  $|\beta_{ij}\rangle$  as follows:

$$\begin{aligned} |00\rangle &= \frac{1}{\sqrt{2}}(|\beta_{00}\rangle + |\beta_{10}\rangle), & |11\rangle &= \frac{1}{\sqrt{2}}(|\beta_{00}\rangle - |\beta_{10}\rangle), \\ |01\rangle &= \frac{1}{\sqrt{2}}(|\beta_{01}\rangle + |\beta_{11}\rangle), & |10\rangle &= \frac{1}{\sqrt{2}}(|\beta_{01}\rangle - |\beta_{11}\rangle). \end{aligned} \quad (2.2)$$

**2.2. Description of quantum measurement. Axiom of quantum measurement:** Quantum measurements are described by a collection  $\{M_m\}$  of projection operators which appear in the spectral decomposition of the observable

$$M = \sum_m m M_m.$$

The index  $m$  refers to the measurement outcomes that may occur in the experiment. If the state of the quantum system is  $|\psi\rangle$  ( $\| |\psi\rangle \| = 1$ ) immediately before the measurement then the probability that result  $m$  occurs is given by

$$p(m) = \langle \psi | M_m | \psi \rangle,$$

and the state of the system after such an ideal measurement is

$$\frac{M_m |\psi\rangle}{\sqrt{\langle \psi | M_m | \psi \rangle}}.$$

The projection operators satisfy the completeness equation,

$$\sum_m M_m = I = \text{Identity operator on the state space.}$$

**2.3. Quantum teleportation.** Now we can describe the process of quantum teleportation, which is illustrated in Fig. 1.

(1) Alice and Bob prepare an EPR pair  $|\beta_{00}\rangle_{AB}$

$$|\beta_{00}\rangle_{AB} = \frac{1}{\sqrt{2}}(|0\rangle_A \otimes |0\rangle_B + |1\rangle_A \otimes |1\rangle_B) = \frac{1}{\sqrt{2}}(|0\rangle_A |0\rangle_B + |1\rangle_A |1\rangle_B)$$

by applying a Hadamard gate and a controlled-not gate to the state  $|0\rangle_A \otimes |0\rangle_B$ , the composite state of Alice's state  $|0\rangle_A$  and

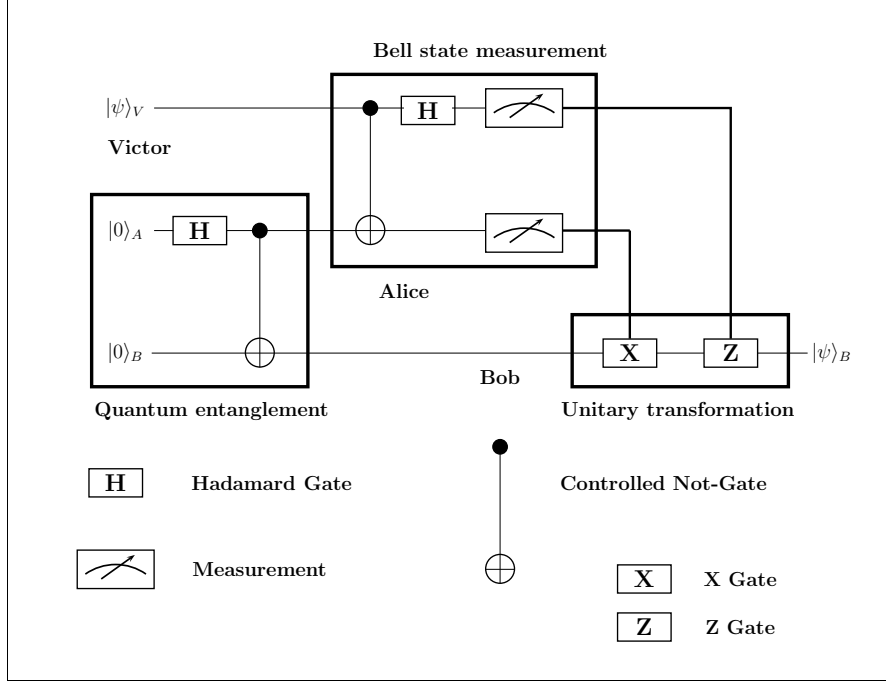


FIGURE 1. Quantum circuit of teleportation. Victor's state  $|\psi\rangle_V$  is given to Alice and reproduced at Bob's laboratory.

Bob's state  $|0\rangle_B$ . Now Alice and Bob share the state  $|\beta_{00}\rangle_{AB}$  ( $|0\rangle_A, |1\rangle_A$  are Alice's states, and  $|0\rangle_B, |1\rangle_B$  are Bob's states). Victor gives Alice a state  $|\psi\rangle_V$  ( $\| |\psi\rangle_V \| = 1$ ) to send to Bob. The state of the total system is

$$|\psi_0\rangle = |\psi\rangle_V \otimes |\beta_{00}\rangle_{AB} = (\alpha|0\rangle_V + \beta|1\rangle_V) \otimes \frac{1}{\sqrt{2}}(|0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B).$$

The state  $|\psi_0\rangle = |\psi\rangle_V \otimes |\beta_{00}\rangle_{AB}$  can be rewritten as:

$$\begin{aligned}
 |\psi_0\rangle &= (\alpha|0\rangle_V + \beta|1\rangle_V) \otimes \frac{1}{\sqrt{2}}(|0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B) \\
 &= \frac{1}{\sqrt{2}}[\alpha|0\rangle_V|0\rangle_A|0\rangle_B + \alpha|0\rangle_V|1\rangle_A|1\rangle_B \\
 &\quad + \beta|1\rangle_V|0\rangle_A|0\rangle_B + \beta|1\rangle_V|1\rangle_A|1\rangle_B] \\
 &= \frac{1}{2}[\alpha(|\beta_{00}\rangle_{VA} + |\beta_{10}\rangle_{VA})|0\rangle_B + \alpha(|\beta_{01}\rangle_{VA} + |\beta_{11}\rangle_{VA})|1\rangle_B \\
 &\quad + \beta(|\beta_{01}\rangle_{VA} - |\beta_{11}\rangle_{VA})|0\rangle_B + \beta(|\beta_{00}\rangle_{VA} - |\beta_{10}\rangle_{VA})|1\rangle_B] \\
 &= \frac{1}{2}[|\beta_{00}\rangle_{VA}(\alpha|0\rangle_B + \beta|1\rangle_B) + |\beta_{01}\rangle_{VA}(\alpha|1\rangle_B + \beta|0\rangle_B) \\
 (2.3) \quad &\quad + |\beta_{10}\rangle_{VA}(\alpha|0\rangle_B - \beta|1\rangle_B) + |\beta_{11}\rangle_{VA}(\alpha|1\rangle_B - \beta|0\rangle_B)].
 \end{aligned}$$

- (2) Alice performs the Bell-state measurement, a measurement which determines  $|\beta_{ij}\rangle$ , and the state of the system after the measurement is

$$\begin{cases} |\beta_{00}\rangle_{VA}(\alpha|0\rangle_B + \beta|1\rangle_B) & \text{if } (ij) = (00) \\ |\beta_{01}\rangle_{VA}(\alpha|1\rangle_B + \beta|0\rangle_B) & \text{if } (ij) = (01) \\ |\beta_{10}\rangle_{VA}(\alpha|0\rangle_B - \beta|1\rangle_B) & \text{if } (ij) = (10) \\ |\beta_{11}\rangle_{VA}(\alpha|1\rangle_B - \beta|0\rangle_B) & \text{if } (ij) = (11) \end{cases}.$$

- (3) Alice sends the classical information  $(ij)$  to Bob. Then Bob sends Bob's qubit through  $I, X, Z, XZ$  according to the result  $(00), (01), (10), (11)$ , obtaining  $\alpha|0\rangle_B + \beta|1\rangle_B$ .

- (4) Since the operator  $(H \otimes I)M_{\text{CNOT}}$  is the inverse of the operator  $M_{\text{CNOT}}(H \otimes I)$ ,  $(H \otimes I)M_{\text{CNOT}}$  sends the Bell basis  $|\beta_{ij}\rangle_{VA}$  to the canonical basis  $|ij\rangle_{VA} = |i\rangle_V \otimes |j\rangle_A$  of product states. Thus Alice's Bell-state measurement is performed by sending the Bell basis to the canonical basis  $|ij\rangle_{VA}$  and making the measurement determining  $|ij\rangle_{VA}$ . In practice, Alice sends her qubits through a CNOT gate, and then the first qubit through a Hadamard gate  $H$ , obtaining

$$(2.4) \quad \frac{1}{2} [ |00\rangle_{VA}(\alpha|0\rangle_B + \beta|1\rangle_B) + |01\rangle_{VA}(\alpha|1\rangle_B + \beta|0\rangle_B) \\ + |10\rangle_{VA}(\alpha|0\rangle_B - \beta|1\rangle_B) + |11\rangle_{VA}(\alpha|1\rangle_B - \beta|0\rangle_B) ].$$

Let  $M_{ij} = |ij\rangle_{VA} \langle ij| \otimes I_B$ ,  $(i, j = 1, 2)$ . Then these projection operators satisfy the completeness equation.

- (5) Alice performs a measurement  $\{M_{ij}\}$ , i.e., measures the observable  $M = \sum_{i,j=0}^1 (2i + j)M_{ij}$ . The probability that result  $(ij)$  occurs (equivalently  $m = 2i + j$ ) is

$$p(ij) = \langle \psi_2 | M_{ij} | \psi_2 \rangle = \frac{1}{4},$$

and the state of the system after the measurement is

$$\frac{M_{ij}|\psi_2\rangle}{\sqrt{\langle \psi_2 | M_{ij} | \psi_2 \rangle}} = \begin{cases} |00\rangle_{VA}(\alpha|0\rangle_B + \beta|1\rangle_B) & \text{if } (ij) = (00) & m = 0 \\ |01\rangle_{VA}(\alpha|1\rangle_B + \beta|0\rangle_B) & \text{if } (ij) = (01) & m = 1 \\ |10\rangle_{VA}(\alpha|0\rangle_B - \beta|1\rangle_B) & \text{if } (ij) = (10) & m = 2 \\ |11\rangle_{VA}(\alpha|1\rangle_B - \beta|0\rangle_B) & \text{if } (ij) = (11) & m = 3 \end{cases}.$$

### 3. EXPERIMENT OF THE QUBIT TELEPORTATION

In this section we discuss the experiment performed by Bouwmeester et al. in 1997 [4] using qubit states. Two pairs of entangled photons are generated by a polarized non-degenerate parametric process. Let  $|0\rangle$  represent the horizontally polarized single photon state  $|\leftrightarrow\rangle$  and  $|1\rangle$  the vertically polarized single photon state  $|\updownarrow\rangle$ .

For these polarization states we calculate the Bell basis according to (2.1). Then the state  $|\psi_0\rangle = |\psi\rangle_V \otimes |\beta_{00}\rangle_{AB}$  is rewritten as follows, (see (2.3)):

$$|\psi_0\rangle = \frac{1}{2} [|\beta_{00}\rangle_{VA}(\alpha|0\rangle_B + \beta|1\rangle_B) + |\beta_{01}\rangle_{VA}(\alpha|1\rangle_B + \beta|0\rangle_B) \\ + |\beta_{10}\rangle_{VA}(\alpha|0\rangle_B - \beta|1\rangle_B) + |\beta_{11}\rangle_{VA}(\alpha|1\rangle_B - \beta|0\rangle_B)].$$

A Bell-state measurement is possible for the state  $|\beta_{11}\rangle_{VA}$ , under the condition that there are at most one photon on each mode of Victor and Alice. Let  $a_{Vj}, a_{Ak}$  be annihilation operators of polarization  $j, k = \leftrightarrow, \uparrow, \downarrow$ . The half-beam splitter causes the Bogoliubov transformation (see (5.4))

$$b_{0j} = \frac{1}{\sqrt{2}}(a_{Vj} + a_{Aj}), \quad b_{1j} = \frac{1}{\sqrt{2}}(-a_{Vj} + a_{Aj}).$$

Let  $|\Omega\rangle$  be the vacuum for  $a_{Vj}, a_{Ak}$ . Then we have

$$\langle\Omega|b_{1k}b_{0j} = \frac{1}{2}\langle\Omega|(-a_{Vk} + a_{Ak})(a_{Vj} + a_{Aj}).$$

We assume that there is at most one photon in each mode (Victor's mode or Alice's mode). On such a condition, we can perform the Bell-state measurement by the simultaneous photon counting after the half-beam splitter. For such a state  $|\psi\rangle$ , we can ignore  $a_{Vk}a_{Vj}$  and  $a_{Ak}a_{Aj}$  we have

$$\langle\Omega|b_{1k}b_{0j}|\psi\rangle = \langle\Omega|(-a_{Aj}a_{Vk} + a_{Ak}a_{Vj})|\psi\rangle = \pm \frac{1}{\sqrt{2}} \langle\beta_{11}|\psi\rangle$$

for  $j \neq k$ , and  $\langle\Omega|b_{1k}b_{0j}|\psi\rangle = 0$  for  $j = k$ , where we used the fact

$$(-a_{Vk}^\dagger a_{Aj}^\dagger + a_{Vj}^\dagger a_{Ak}^\dagger)|\Omega\rangle = \pm \frac{1}{\sqrt{2}}|\beta_{11}\rangle_{VA}.$$

This shows that the simultaneous photon detection is equivalent to  ${}_{VA}\langle\beta_{11}|$ , that is, equivalent to the Bell-state measurement. However, this measurement cannot identify the other three Bell states, and has a fatal drawback. The photon counting technique of today cannot distinguish whether only one photon is coming or more than two photons are coming simultaneously. So, if two photons come in Victor's mode simultaneously, we cannot neglect the term  $a_{Vk}a_{Vj}$ , the measurement is not the Bell-state measurement.

#### 4. HOLOMORPHIC REPRESENTATION OF CCR

The “holomorphic representation” of the canonical commutation relations (CCR) was introduced by Bargmann [1] for the finite dimensional case. In the case of infinitely many degrees of freedom it was introduced by Segal [14] (see also [15]). One of the most famous applications of this representation we find in the book [3] by Berezin, where it has the interesting counterpart to the Fermion case, i.e., canonical anti-commutation relations. The canonical anti-commutation relation

has a representation which is similar to the holomorphic representation where however the field of complex numbers is replaced by the Grassmann algebra. Such a representation is now popular under the name of Berezin calculus. Another important application of the holomorphic representation is given in the book of Faddeev and Slanov [8].

#### 4.1. The holomorphic (Bargmann) representation of the CCR.

We develop the representation of canonical commutation relation (CCR) called holomorphic representation, which seems to be quite useful for quantum optics and quantum teleportation of continuous variables.

The operator of multiplication  $q$  and the differentiation  $p = -id/dq$  in  $L^2(\mathbb{R}) = L^2(\mathbb{R}, dq)$  satisfies the commutation relation:

$$[q, p] = qp - pq = iI$$

on a suitable subspace of  $L^2(\mathbb{R})$ . Introduce the operators

$$a = (q + d/dq)/\sqrt{2} = (q + ip)/\sqrt{2}, \quad a^\dagger = (q - ip)/\sqrt{2}$$

then

$$[a, a^\dagger] = 1, \quad q = (a + a^\dagger)/\sqrt{2}, \quad p = (a - a^\dagger)/\sqrt{2}i.$$

This representation of the commutation relation is called the *Schrödinger representation*. The function  $f(q) = e^{-q^2/2}$  is a solution of the equation

$$0 = \sqrt{2}af(q) = (q + d/dq)f(q).$$

We consider the space  $L^2(\mathbb{R}, e^{-q^2}dq/\sqrt{\pi})$  and the unitary operator  $U$

$$U : L^2(\mathbb{R}, e^{-q^2}dq/\sqrt{2\pi}) \ni g(q) \rightarrow (\pi)^{-1/2}g(q)e^{-q^2/2} \in L^2(\mathbb{R}, dq).$$

Then we have

$$U^\dagger q U = q, \quad U^\dagger (d/dq) U = d/dq - q,$$

$$b = U^\dagger a U = U^\dagger (q + d/dq) U / \sqrt{2} = 2^{-1/2} d/dq,$$

$$b^\dagger = U^\dagger a^\dagger U = U^\dagger (q - d/dq) U / \sqrt{2} = 2^{-1/2} (2q - d/dq).$$

This representation  $L^2(\mathbb{R}, e^{-q^2}dq/\sqrt{2\pi})$ ,  $b = 2^{-1/2}d/dq$ ,  $b^\dagger = 2^{-1/2}(2q - d/dq)$  is called the *modified Schrödinger representation*.

Next we consider

$$u = x + iy, \quad L^2(\mathbb{C}, d\mu), \quad d\mu = e^{-\bar{u}u} \frac{d\bar{u}du}{2\pi i} = e^{-(x^2+y^2)} \frac{dx dy}{\pi}, \quad \int_{\mathbb{C}} d\mu = 1,$$

and its subspace  $\mathcal{H}$  generated by holomorphic functions of  $\bar{u}$  (anti-holomorphic functions). For  $f, g \in \mathcal{H}$  define the inner product by

$$\langle f | g \rangle = \int \overline{f(\bar{u})} g(\bar{u}) e^{-\bar{u}u} \frac{d\bar{u}du}{2\pi i}.$$

Then the multiplication operator  $f(\bar{u}) \rightarrow \bar{u}f(\bar{u})$  and the differential operator  $f(\bar{u}) \rightarrow \partial/\partial \bar{u} f(\bar{u})$  are adjoint to each other:

$$\int \overline{f(\bar{u})} \{ \partial/\partial \bar{u} g(\bar{u}) \} e^{-\bar{u}u} \frac{d\bar{u}du}{2\pi i} = - \int \partial/\partial \bar{u} \{ e^{-\bar{u}u} \overline{f(\bar{u})} \} g(\bar{u}) \frac{d\bar{u}du}{2\pi i}$$



$$= \int \overline{f(\bar{u})} u g(\bar{u}) e^{-\bar{u}u} \frac{d\bar{u}du}{2\pi i} = \int \bar{u} \overline{f(\bar{u})} g(\bar{u}) e^{-\bar{u}u} \frac{d\bar{u}du}{2\pi i},$$

where we used the relation  $\partial/\partial \bar{u} \overline{f(\bar{u})} = 0$  because  $\overline{f(\bar{u})}$  is a holomorphic function of  $u$ . If we put

$$a = \partial/\partial \bar{u} = (1/2)(\partial/\partial x + i\partial/\partial y), \quad a^\dagger = \bar{u},$$

then  $a$  and  $a^\dagger$  satisfy the commutation relation  $[a, a^\dagger] = 1$ . We call this representation the *holomorphic representation*. In this representation,  $\{\bar{u}^n/\sqrt{n!}\}_{n=0}^\infty$  is an orthonormal basis. In fact, let  $m \leq n$ . Then we have

$$\begin{aligned} \int \frac{u^m}{\sqrt{m!}} \frac{\bar{u}^n}{\sqrt{n!}} e^{-\bar{u}u} \frac{d\bar{u}du}{2\pi i} &= \int \frac{u^m}{\sqrt{m!}} \left( \frac{(-1)^n \partial^n}{\sqrt{n!} \partial u^n} \right) e^{-\bar{u}u} \frac{d\bar{u}du}{2\pi i} \\ &= \int \left( \frac{\partial^n}{\sqrt{n!} \partial u^n} \frac{u^m}{\sqrt{m!}} \right) e^{-\bar{u}u} \frac{d\bar{u}du}{2\pi i} = \delta_{mn}, \end{aligned}$$

and if  $m > n$ ,

$$\int \frac{u^m}{\sqrt{m!}} \frac{\bar{u}^n}{\sqrt{n!}} e^{-\bar{u}u} \frac{d\bar{u}du}{2\pi i} = \int \left( \frac{\partial^m}{\sqrt{m!} \partial u^m} \frac{\bar{u}^n}{\sqrt{n!}} \right) e^{-\bar{u}u} \frac{d\bar{u}du}{2\pi i} = 0.$$

Therefore, if  $f(\bar{u}) = \sum_{n=0}^\infty a_n \bar{u}^n$  then  $\|f\|^2 = \sum_{n=0}^\infty |a_n|^2 n!$ . The coherent state is an eigen state of the annihilation operator. Also in the holomorphic representation,

$$\frac{\partial}{\partial \bar{u}} f(\bar{u}) = \alpha f(\bar{u}), \quad f(\bar{u}) = C e^{\alpha \bar{u}} = C \sum_{n=0}^\infty \frac{(\alpha \bar{u})^n}{n!},$$

$$C = e^{-|\alpha|^2/2} \Rightarrow \|f\|^2 = |C|^2 \sum_{n=0}^\infty \frac{|\alpha|^{2n}}{n!} = |C|^2 e^{|\alpha|^2} = 1.$$

Hence,  $f$  is normalized by choosing  $C = e^{-|\alpha|^2/2}$ . Furthermore, in the holomorphic representation, the integral kernel  $K(\bar{u}, v)$  of the identity operator is

$$K(\bar{u}, v) = e^{\bar{u}v}.$$

In fact, for  $\alpha \in \mathbb{C}$ ,

$$\int e^{\alpha v} \bar{v}^n e^{-\bar{v}v} \frac{d\bar{v}dv}{2\pi i} = \int \sum_{k=0}^\infty \frac{(\alpha v)^k}{k!} \bar{v}^n e^{-\bar{v}v} \frac{d\bar{v}dv}{2\pi i} = \alpha^n$$

implies

(4.1)

$$\int e^{\alpha v} f(\bar{v}) e^{-\bar{v}v} \frac{d\bar{v}dv}{2\pi i} = \int e^{\alpha v} \sum_{n=0}^\infty a_n \bar{v}^n e^{-\bar{v}v} \frac{d\bar{v}dv}{2\pi i} = \sum_{n=0}^\infty a_n \alpha^n = f(\alpha).$$

Because of symmetry in  $v$  and  $\bar{v}$ , one also has

$$(4.2) \quad \int e^{\alpha \bar{v}} f(v) e^{-\bar{v}v} \frac{d\bar{v}dv}{2\pi i} = f(\alpha).$$

The equality

$$a^{\dagger m} a^n f(\bar{u}) = \bar{u}^m \frac{\partial^n}{\partial \bar{u}^n} \int e^{\bar{u}v} f(v) e^{-\bar{v}v} \frac{d\bar{v}dv}{2\pi i} = \int \bar{u}^m v^n e^{\bar{u}v} f(v) e^{-\bar{v}v} \frac{d\bar{v}dv}{2\pi i}$$

shows that the integral kernel of the normal ordered monomial  $a^{\dagger m} a^n$  of  $a^{\dagger}$  and  $a$  is  $\bar{u}^m v^n e^{\bar{u}v}$ .

The holomorphic representation for the case of  $n$  degrees of freedom is briefly described below:

The variables, the Hilbert space and the measure now are:

$$\bar{u} = (\bar{u}_1, \dots, \bar{u}_n), \quad L^2(\mathbb{C}^n, d\mu_n), \quad d\mu_n = \prod_{j=1}^n e^{-\bar{u}_j u_j} \frac{d\bar{u}_j du_j}{2\pi i}.$$

**4.2. Integral kernels of basic operations in the holomorphic representation.** For the mathematical description of the devices for the manipulation of photon states we need several results for the kernels of various linear transformations of the basic creation and annihilation operators. The relevant results are stated in this subsection. The proofs are contained in the appendix.

These results are special cases of results given in [3] where the infinite dimensional counter part is given. The proofs of these results in [3] are based on the use of functional integration. Though the basic strategy is the same as in [3], the proofs given in our appendix only uses elementary mathematical tools so that these result become more easily accessible.

**Theorem 4.1.** *The linear canonical transformation*

$$(4.3) \quad b_j = a_j + f_j, \quad b_j^{\dagger} = a_j^{\dagger} + \bar{f}_j$$

is implemented by the unitary operator  $U$  whose integral kernel  $U(\bar{u}, v)$  is

$$U(\bar{u}, v) = c \exp \sum_{j=1}^n (\bar{u}_j v_j + v_j \bar{f}_j - \bar{u}_j f_j), \quad c = \theta \exp \left\{ (-1/2) \sum_{j=1}^n \bar{f}_j f_j \right\}, \quad |\theta| = 1.$$

**Theorem 4.2.** *The linear canonical transformation*

$$b_j = \sum_{k=1}^n (\Phi_{jk} a_k + \Psi_{jk} a_k^{\dagger}), \quad b_j^{\dagger} = \sum_{k=1}^n (\bar{\Phi}_{jk} a_k^{\dagger} + \bar{\Psi}_{jk} a_k)$$

is implemented by the unitary operator  $U$  whose integral kernel  $U(\bar{u}, v)$  is given by

$$U(\bar{u}, v) = c \exp \left( \frac{1}{2} (v \ \bar{u}) \begin{pmatrix} A^{11} & A^{12} \\ A^{21} & A^{22} \end{pmatrix} \begin{pmatrix} v \\ \bar{u} \end{pmatrix} \right), \quad A^{jk} = {}^t A^{kj},$$

$$A^{22} = -\Phi^{-1} \Psi, \quad A^{21} = \Phi^{-1}, \quad A^{11} = \bar{\Psi} \Phi^{-1}, \quad c = \theta (\det \Phi \Phi^{\dagger})^{-1/4}, \quad |\theta| = 1.$$

**Theorem 4.3.** *Let  $f_j$  be complex numbers and*

$$H = \sum_{j=1}^n (f_j a_j^\dagger + \bar{f}_j a_j)$$

*be a self-adjoint operator. Then the kernel function  $U(\bar{u}, v)$  of  $e^{itH}$  is given by*

(4.4)

$$U(\bar{u}, v) = c \exp \sum_{j=1}^n (\bar{u}_j v_j + it \bar{f}_j v_j + it f_j \bar{u}_j), \quad c = \exp \left\{ -\frac{1}{2} t^2 \sum_{j=1}^n \bar{f}_j f_j \right\}.$$

**Theorem 4.4.** *Let  $B$  and  $C$  be  $n \times n$  matrices, and*

$$H = \frac{1}{2} (a^\dagger B a^\dagger + a \bar{B} a + 2a^\dagger C a)$$

*be a self-adjoint operator. Further, denote*

$$\mathcal{A} = \begin{pmatrix} -C & -B \\ \bar{B} & \bar{C} \end{pmatrix}, \quad e^{it\mathcal{A}} = \begin{pmatrix} \Phi & \Psi \\ \bar{\Psi} & \bar{\Phi} \end{pmatrix}.$$

*Then the kernel function  $U(\bar{u}, v)$  of  $e^{itH}$  is given by*

$$(4.5) \quad U(\bar{u}, v) = c \exp \left( \frac{1}{2} (v \ \bar{u}) \begin{pmatrix} A^{11} & A^{12} \\ A^{21} & A^{22} \end{pmatrix} \begin{pmatrix} v \\ \bar{u} \end{pmatrix} \right), \quad A^{jk} = {}^t A^{kj}$$

*with*

$$(4.6) \quad A^{22} = -\Phi^{-1} \Psi, \quad A^{21} = \Phi^{-1}, \quad A^{11} = \bar{\Psi} \Phi^{-1}, \quad c = (\det \Phi e^{itC})^{-1/2}.$$

## 5. PHOTON STATES AND DEVICES FOR MANIPULATING PHOTON STATES

We consider a self-adjoint operator

$$H_{\text{laser}} = i(\bar{\alpha} a - \alpha a^\dagger)$$

and an operator  $D(\alpha) = e^{iH_{\text{laser}}(\alpha)}$  called the displacement operator which generates from the vacuum state a state called the *coherent state* which is considered to represent the laser-beam state. In order to have the kernel  $U(\bar{u}, v)$  of the operator  $e^{iH_{\text{laser}}(g)}$ , we use Theorem 4.3, i.e.,  $n = 1$  and  $f = -i\alpha$ . The kernel  $U(\bar{u}, v)$  of the operator  $D(\alpha)$  therefore is

$$U(\bar{u}, v) = e^{-|\alpha|^2/2} \exp\{\bar{u}v - \bar{\alpha}v + \alpha\bar{u}\}.$$

In the holomorphic representation, its action on a state can be calculated explicitly

$$\begin{aligned} (D(\alpha)f)(\bar{u}) &= \int U(\bar{u}, v) f(v) e^{-\bar{v}v} \frac{d\bar{v}dv}{2\pi i} \\ &= e^{-|\alpha|^2/2} \int \exp\{\bar{u}v - \bar{\alpha}v + \alpha\bar{u}\} f(v) e^{-\bar{v}v} \frac{d\bar{v}dv}{2\pi i} \\ &= e^{-|\alpha|^2/2} e^{\alpha\bar{u}} \int \exp\{(\bar{u} - \bar{\alpha})v\} f(v) e^{-\bar{v}v} \frac{d\bar{v}dv}{2\pi i} \end{aligned}$$

$$(5.1) \quad = e^{-|\alpha|^2/2} e^{\alpha \bar{u}} f(\bar{u} - \bar{\alpha}).$$

If  $f(\bar{u}) = 1$  which corresponds to the vacuum  $|0\rangle$ , then

$$|\alpha\rangle = D(\alpha)|0\rangle = e^{-|\alpha|^2/2} e^{\alpha \bar{u}}$$

is the coherent state and satisfies the following completeness relation:

$$(5.2) \quad \int |\alpha\rangle\langle\alpha| \frac{d\bar{\alpha}d\alpha}{2\pi i} = \int e^{-\bar{\alpha}\alpha} e^{\alpha \bar{u}} e^{\bar{\alpha}v} \frac{d\bar{\alpha}d\alpha}{2\pi i} = e^{\bar{u}v},$$

where we used the fact that  $e^{\bar{u}v}$  is the kernel of the identity operator  $I$ . Next consider a self-adjoint operator  $H_{\text{para}}(g) = ig(a^2 - a^{\dagger 2})$ , the generator of parametric amplification. In order to have the kernel  $U(\bar{u}, v)$  of the operator  $e^{iH_{\text{para}}(g)}$ , we use Theorem 4.4, i.e.,  $B = -ig$  and  $C = 0$ , and

$$\begin{aligned} \mathcal{A} &= \begin{pmatrix} 0 & ig \\ ig & 0 \end{pmatrix}, \quad \begin{pmatrix} \Phi & \Psi \\ \bar{\Psi} & \bar{\Phi} \end{pmatrix} = e^{it\mathcal{A}} = \exp tg \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \\ &= \cosh tg \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \sinh tg \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} \cosh tg & -\sinh tg \\ -\sinh tg & \cosh tg \end{pmatrix}, \\ A^{22} &= -\Phi^{-1}\Psi = \tanh tg, \quad A^{21} = \Phi^{-1} = \cosh^{-1} tg, \quad A^{11} = \bar{\Psi}\Phi^{-1} = -\tanh tg. \end{aligned}$$

The kernel  $U(\bar{u}, v)$  of the operator  $e^{iH_{\text{para}}(g)}$  therefore is

$$U(\bar{u}, v) = (\cosh g)^{-1/2} \exp(1/2) \{ \tanh g \bar{u}^2 + 2 \cosh^{-1} g \bar{u}v - \tanh g v^2 \}.$$

In the holomorphic representation, its action on the vacuum state can be calculated explicitly

$$\begin{aligned} e^{iH_{\text{para}}(g)}|0\rangle &= \int U(\bar{u}, v) e^{-\bar{v}v} \frac{d\bar{v}dv}{2\pi i} \\ &= (\cosh g)^{-1/2} \exp(1/2) \{ \tanh g \bar{u}^2 \} \\ &\quad \times \int \exp(1/2) \{ 2 \cosh^{-1} g \bar{u}v - \tanh g v^2 \} e^{-\bar{v}v} \frac{d\bar{v}dv}{2\pi i} \\ (5.3) \quad &= (1 - \tanh^2 g)^{1/4} \exp(1/2) \{ \tanh g \bar{u}^2 \} \end{aligned}$$

where we used the relation (4.2). The state (5.3) is called the squeezed vacuum with squeezing parameter  $g$ .

Let us consider the beam splitter. The generator of the beam splitter  $H_{\text{bs}}(\theta)$  is defined by

$$H_{\text{bs}}(\theta) = i\theta(a_1^\dagger a_2 - a_1 a_2^\dagger).$$

Then applying the Theorem 4.4 again, we get the kernel  $U(\bar{u}, v)$  of  $e^{iH_{\text{bs}}(\theta)}$  as follows

$$B = 0, \quad C = \begin{pmatrix} 0 & i\theta \\ -i\theta & 0 \end{pmatrix}, \quad \mathcal{A} = \begin{pmatrix} -C & 0 \\ 0 & \bar{C} \end{pmatrix},$$

$$\begin{aligned}
\begin{pmatrix} \Phi & \Psi \\ \bar{\Psi} & \bar{\Phi} \end{pmatrix} &= e^{itA} = \begin{pmatrix} e^{-iC} & 0 \\ 0 & e^{it\bar{C}} \end{pmatrix}, \\
e^{-itC} = e^{it\bar{C}} &= \exp t\theta \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \cos t\theta \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \sin t\theta \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\
\Phi &= \begin{pmatrix} \cos t\theta & \sin t\theta \\ -\sin t\theta & \cos t\theta \end{pmatrix}, \quad \Psi = 0. \\
U(\bar{u}, v) &= c \exp \left( \frac{1}{2} (v \ \bar{u}) \begin{pmatrix} A^{11} & A^{12} \\ A^{21} & A^{22} \end{pmatrix} \begin{pmatrix} v \\ \bar{u} \end{pmatrix} \right), \quad A^{jk} = {}^t A^{kj} \\
A^{22} = A^{11} = 0, \quad A^{21} = \Phi^{-1} &= \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}, \quad c = (\det \Phi e^{iC})^{-1/2} = 1.
\end{aligned}$$

The beam splitter for  $\theta = \pi/4$  is called the *half-beam splitter* whose generator is  $H_{hbs} = i(\pi/4)(a_1^\dagger a_2 - a_1 a_2^\dagger)$ . The half-beam splitter transforms the state  $f(\bar{u}_1, \bar{u}_2)$  into

$$\begin{aligned}
&\int U(\bar{u}, v) f(\bar{v}_1, \bar{v}_2) e^{-\bar{v}v} \prod_{j=1}^2 \frac{d\bar{v}_j dv_j}{2\pi i} \\
&\int \exp \frac{1}{\sqrt{2}} \{ (\bar{u}_1 + \bar{u}_2)v_1 + (-\bar{u}_1 + \bar{u}_2)v_2 \} f(\bar{v}_1, \bar{v}_2) e^{-\bar{v}v} \prod_{j=1}^2 \frac{d\bar{v}_j dv_j}{2\pi i} \\
(5.4) \quad &= f((\bar{u}_1 + \bar{u}_2)/\sqrt{2}, (-\bar{u}_1 + \bar{u}_2)/\sqrt{2}).
\end{aligned}$$

**5.1. Balanced homodyne detection.** We calculate the mean value of the difference of photon beams  $N_1 - N_2 = a_1^\dagger a_1 - a_2^\dagger a_2$  for the state  $e^{-iH_{hbs}}|\psi_1\rangle \otimes |\psi_2\rangle$ , the state after passing through the half-beam splitter. Since

$$\begin{aligned}
e^{iH_{hbs}} a_1 e^{-iH_{hbs}} &= (a_1 + a_2)/\sqrt{2}, \quad e^{iH_{hbs}} a_2 e^{-iH_{hbs}} = (-a_1 + a_2)/\sqrt{2}, \\
\langle \psi_1 | \otimes \langle \psi_2 | e^{iH_{hbs}} (N_1 - N_2) e^{-iH_{hbs}} | \psi_1 \rangle \otimes | \psi_2 \rangle \\
&= \langle \psi_1 | \otimes \langle \psi_2 | (a_1^\dagger a_2 + a_1 a_2^\dagger) | \psi_1 \rangle \otimes | \psi_2 \rangle.
\end{aligned}$$

If  $|\psi_2\rangle$  is a coherent state  $|\alpha_2\rangle$ , then this identity is continued by

$$\begin{aligned}
&= \langle \psi_1 | \otimes \langle \alpha_2 | (a_1^\dagger a_2 + a_1 a_2^\dagger) | \psi_1 \rangle \otimes | \alpha_2 \rangle = \langle \psi_1 | a_1^\dagger | \psi_1 \rangle \alpha_2 + \langle \psi_1 | a_1 | \psi_1 \rangle \bar{\alpha}_2 \\
&= \langle \psi_1 | (q_1 - ip_1)/\sqrt{2} | \psi_1 \rangle | \alpha_2 \rangle e^{i\theta_2} + \langle \psi_1 | (q_1 + ip_1)/\sqrt{2} | \psi_1 \rangle | \alpha_2 \rangle e^{-i\theta_2} \\
&= \sqrt{2} \langle \psi_1 | q_1 | \psi_1 \rangle | \alpha_2 \rangle \cos \theta_2 + \sqrt{2} \langle \psi_1 | p_1 | \psi_1 \rangle | \alpha_2 \rangle \sin \theta_2 \\
&= \sqrt{2} | \alpha_2 \rangle \langle \psi_1 | (q_1 \cos \theta_2 + p_1 \sin \theta_2) | \psi_1 \rangle.
\end{aligned}$$

Thus the balanced homodyne detection measures  $\sqrt{2}|\alpha_2|$  times of  $q_1$  and  $p_2$  according to the phase of the coherent beam  $\alpha_2$ .

The displacement operator  $D(-\alpha)$  is realized by modulators and the beam splitter. For  $|\phi\rangle \in \mathcal{H}_2$  and  $|\beta\rangle \in \mathcal{H}_3$ , we send  $|\phi\rangle \otimes |\beta\rangle$  through a beam splitter  $B(\theta) = e^{iH_{bs}(\theta)}$ . Let

$$|\phi\rangle \otimes |\beta\rangle = \phi(\bar{u}_2) e^{-|\beta|^2/2} e^{\beta \bar{u}_3}.$$

Then

$$\begin{aligned}
B(\theta)(|\phi\rangle \otimes |\beta\rangle) &= B(\theta)\phi(\bar{u}_2)e^{-|\beta|^2/2}e^{\beta\bar{u}_3} \\
&= \phi(\bar{u}_2 \cos \theta + \bar{u}_3 \sin \theta)e^{-|\beta|^2/2}e^{\beta(-\bar{u}_2 \sin \theta + \bar{u}_3 \cos \theta)} \\
&= \phi(\bar{u}_2 \cos \theta + \bar{u}_3 \sin \theta)e^{-|\beta|^2/2}e^{-\beta\bar{u}_2 \sin \theta}e^{\beta\bar{u}_3 \cos \theta} \\
&\quad (I \otimes \langle \beta |)B(\theta)|\phi\rangle \otimes |\beta\rangle \\
&= \int e^{-|\beta|^2}e^{\beta u_3}\phi(\bar{u}_2 \cos \theta + \bar{u}_3 \sin \theta)e^{-\beta\bar{u}_2 \sin \theta}e^{\beta\bar{u}_3 \cos \theta}e^{-\bar{u}_3 u_3}\frac{d\bar{u}_3 du_3}{2\pi i} \\
&= e^{-|\beta|^2}e^{-\beta\bar{u}_2 \sin \theta} \int e^{\bar{\beta} u_3}e^{\beta\bar{u}_3 \cos \theta}\phi(\bar{u}_2 \cos \theta + \bar{u}_3 \sin \theta)\frac{d\bar{u}_3 du_3}{2\pi i} \\
&= e^{-|\beta|^2}e^{-\beta\bar{u}_2 \sin \theta}e^{\beta\bar{\beta} \cos \theta}\phi(\bar{u}_2 \cos \theta + \bar{\beta} \sin \theta).
\end{aligned}$$

Let  $\theta \rightarrow 0$  and  $\beta \rightarrow \infty$  such that  $\beta \sin \theta \rightarrow \alpha$ . Then

$$|\beta|^2(1 - \cos \theta) \rightarrow |\alpha|^2/2$$

and

$$\begin{aligned}
&e^{-|\beta|^2}e^{-\beta\bar{u}_2 \sin \theta}e^{\beta\bar{\beta} \cos \theta}\phi(\bar{u}_2 \cos \theta + \bar{\beta} \sin \theta) \\
&\rightarrow e^{-|\alpha|^2/2}e^{-\alpha\bar{u}_2}\phi(\bar{u}_2 + \bar{\alpha}) = D(\alpha)|\phi\rangle.
\end{aligned}$$

## 6. TELEPORTATION OF CONTINUOUS QUANTUM VARIABLES

The proposal for continuous variable quantum teleportation was first made by Vaidman [17] in 1994, and then by Braunstein and Kimble [6] in 1998, and experimentally demonstrated by the Caltech group, Furusawa et al. [10] in 1998. Now we present the teleportation of continuous quantum variables in a parallel way as the teleportation of qubits in Section 2. The numbering (1), ..., (4) corresponds to that in Section 2. Figure 2 is similar to the Figure 1 and these figures illustrate the correspondence of the two cases of teleportation.

- (1) First, an entangled state as a counterpart to an EPR pair is produced. We prepare such a state using parametric amplification and a beam splitter. By parametric amplification we create a pair of squeezed vacuum states  $U_{\text{para}}(g)|0\rangle = e^{iH_{\text{para}}(g)}|0\rangle$  and  $U_{\text{para}}(-g)|0\rangle$ . In the holomorphic representation one has (see (5.2)),

$$(1 - q^2)^{-1/4}U_{\text{para}}(g)|0\rangle = e^{q\bar{u}_1^2/2},$$

$$(1 - q^2)^{-1/4}U_{\text{para}}(-g)|0\rangle = e^{-q\bar{u}_2^2/2}, \quad q = \tanh g.$$

Then we send these states through the half-beam splitter (see (5.4)). Now Alice and Bob share the state ( $\bar{u}_1$  is Alice's variable and  $\bar{u}_2$  is Bob's variable)

$$|\Psi_0\rangle = e^{q(\bar{u}_1 + \bar{u}_2)^2/4}e^{-q(-\bar{u}_1 + \bar{u}_2)^2/4} = e^{q\bar{u}_1\bar{u}_2}, \quad q = \tanh g.$$

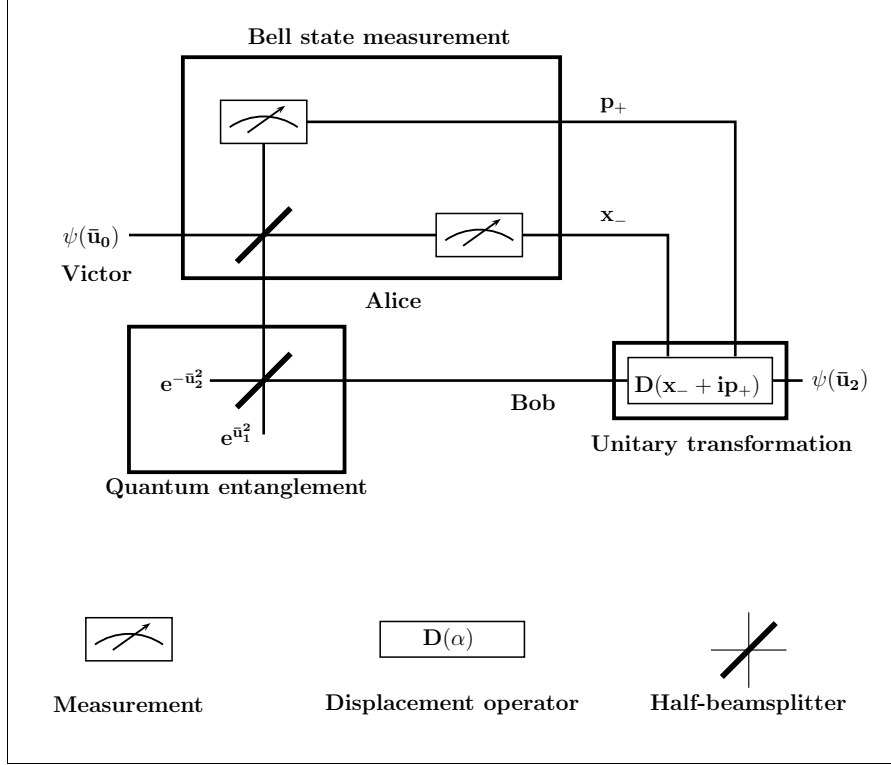


FIGURE 2. Quantum circuit of continuous variable teleportation

Victor gives Alice a state  $\psi(\bar{u}_0)$  to send to Bob. The state of the total system is

$$|\Psi_1\rangle = \psi(\bar{u}_0)e^{q\bar{u}_1\bar{u}_2} = |\psi\rangle_0 \otimes \sum_{n=0}^{\infty} q^n |n\rangle_1 \otimes |n\rangle_2, \quad q = \tanh g.$$

This state corresponds to the state

$$|\psi_0\rangle = |\psi\rangle_V \otimes \frac{1}{\sqrt{2}}(|0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B)$$

of the qubit case. The generalized state (for some background information on generalized states see the first part of the Appendix)

$$\pi^{-1/2}e^{\bar{v}_0\bar{v}_1} = \pi^{-1/2} \sum_{n=0}^{\infty} \frac{(\bar{v}_0\bar{v}_1)^n}{n!} = \pi^{-1/2} \sum_{n=0}^{\infty} |n\rangle_0 \otimes |n\rangle_1$$

corresponds to the Bell state  $|\beta_{00}\rangle = 2^{-1/2} \sum_{n=0}^{\infty} |n\rangle_0 \otimes |n\rangle_1$  of Section 2. The other Bell states  $|\beta_{ij}\rangle$  correspond to the states

$$\sum_{n=0}^{\infty} (D(\alpha)|n\rangle_0) \otimes |n\rangle_1 = e^{-|\alpha|^2/2} e^{(\bar{v}_0 - \bar{\alpha})\bar{v}_1} e^{\alpha\bar{v}_0} = e^{-|\alpha|^2/2} e^{\bar{v}_0\bar{v}_1} e^{-\bar{\alpha}\bar{v}_1} e^{\alpha\bar{v}_0}$$

(see (5.1)).

In the Appendix we show that

$$(6.1) \quad \left\{ \pi^{-1/2} \sum_{n=0}^{\infty} (D(\alpha)|n\rangle_0) \otimes |n\rangle_1; \alpha \in \mathbb{C} \right\}, \quad \alpha = x_- + ip_+$$

is the generalized Bell basis, i.e., a complete orthonormal system in our Hilbert space.

Then we send this state through the half-beam splitter realized as the unitary operator  $e^{iH_{\text{hbs}}}$ , and obtain (see (5.4))

$$e^{-(x_-^2 + p_+^2)/2} e^{-\bar{u}_0^2/2} e^{\bar{u}_1^2/2} e^{\sqrt{2}x_- \bar{u}_0} e^{i\sqrt{2}p_+ \bar{u}_1} = \pi^{1/2} |x_- \rangle \otimes |p_+ \rangle,$$

where

$$|x_- \rangle = \pi^{-1/4} e^{-x_-^2/2} e^{-\bar{u}_0^2/2} e^{\sqrt{2}x_- \bar{u}_0}$$

is the generalized eigen-state of the operator

$$x_0 = \frac{1}{\sqrt{2}} \left( \frac{\partial}{\partial \bar{u}_0} + \bar{u}_0 \right)$$

with eigen-value  $x_-$ , and

$$|p_+ \rangle = \pi^{-1/4} e^{-p_+^2/2} e^{\bar{u}_1^2/2} e^{i\sqrt{2}p_+ \bar{u}_1}$$

is the generalized eigen-state of the operator

$$p_1 = \frac{1}{\sqrt{2}i} \left( \frac{\partial}{\partial \bar{u}_1} - \bar{u}_1 \right)$$

with eigen-value  $p_+$ . In the Appendix the orthogonality relations

$$(6.2) \quad \langle x'_- | x_- \rangle = \delta(x'_- - x_-), \quad \langle p'_+ | p_+ \rangle = \delta(p'_+ - p_+).$$

are shown. Thus we have

$$(6.3) \quad e^{iH_{\text{hbs}}} \sum_{n=0}^{\infty} (D(\alpha)|n\rangle_0) \otimes |n\rangle_1 = \pi^{1/2} |x_- \rangle \otimes |p_+ \rangle$$

and

$$\begin{aligned} & \pi^{-1} \left( \sum_{m=0}^{\infty} {}_0\langle m | (D(\alpha')^\dagger \otimes {}_1\langle m |) \left( \sum_{n=0}^{\infty} (D(\alpha)|n\rangle_0) \otimes |n\rangle_1 \right) \right. \\ & \left. = (\langle x'_- | \otimes \langle p'_+ |), (|x_- \rangle \otimes |p_+ \rangle) = \delta(x'_- - x_-) \delta(p'_+ - p_+). \right. \end{aligned}$$

The state  $|\psi\rangle_V \otimes |\beta_{00}\rangle_{AB}$  corresponds to

$$|\Phi_1\rangle = \pi^{-1/2} |\psi\rangle_0 \otimes \sum_{n=0}^{\infty} q^n |n\rangle_1 \otimes |n\rangle_2.$$

The relation

$$I = \pi^{-1} \int dx_- dp_+ \sum_{m=0}^{\infty} (D(\alpha)|m\rangle_0) \otimes |m\rangle_1 \sum_{k=0}^{\infty} {}_0\langle k | D(\alpha)^\dagger \otimes {}_1\langle k |$$



implies for  $\alpha = x_- + ip_+$  that

$$\begin{aligned}
|\Phi_1\rangle &= \pi^{-3/2} \int dx_- dp_+ \sum_{m=0}^{\infty} (D(\alpha)|m\rangle_0) \otimes |m\rangle_1 \\
&\quad \otimes \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} {}_0\langle k|D(\alpha)^\dagger|\psi\rangle_0 q^n {}_1\langle k|n\rangle_1 |n\rangle_2 \\
&= \pi^{-3/2} \int dx_- dp_+ \sum_{m=0}^{\infty} (D(\alpha)|m\rangle_0) \otimes |m\rangle_1 \otimes \sum_{n=0}^{\infty} q^n {}_0\langle n|D(\alpha)^\dagger|\psi\rangle_0 |n\rangle_2.
\end{aligned}$$

This is the expansion of  $|\Phi_1\rangle$  with respect to the generalized Bell basis (6.1).

- (2) Then Alice performs the (generalized) Bell-state measurement, and when some Bell state  $\pi^{-1/2} \sum_{n=0}^{\infty} (D(\alpha)|n\rangle_0) \otimes |n\rangle_1$  is chosen, then the total system is reduced to

$$= \pi^{-1/2} \sum_{m=0}^{\infty} (D(\alpha)|m\rangle_0) \otimes |m\rangle_1 \otimes \sum_{n=0}^{\infty} q^n {}_0\langle n|D(\alpha)^\dagger|\psi\rangle_0 |n\rangle_2.$$

If  $q = 1$ , the above state is

$$\sum_{m=0}^{\infty} (D(\alpha)|m\rangle_0) \otimes |m\rangle_1 \otimes D(\alpha)^\dagger|\psi\rangle_2.$$

Thus the Bob's state is reduced to  $D(\alpha)^\dagger|\psi\rangle_2$  though he can not know this.

- (3) Alice sends the classical information  $\alpha = x_- + ip_+$  to Bob. Then Bob sends Bob's state  $D(\alpha)^\dagger\psi$  through  $D(\alpha)$  according to the result  $\alpha = x_- + ip_+$ , obtaining  $\psi_{\text{out}} = D(\alpha)D(\alpha)^\dagger\psi = \psi$ . Alice sends two real numbers  $(x_-, p_+)$  to Bob, and Bob gets a state  $\psi = \sum_{n=0}^{\infty} a_n |n\rangle$  in an infinite dimensional Hilbert space.
- (4) The Bell-state measurement (2) is done by sending the Bell basis through the half-beam splitter obtaining the canonical basis  $\{|x_-\rangle \otimes |p_+\rangle; \alpha = x_- + ip_+ \in \mathbb{C}\}$  of (6.3) and by the measurement of  $x_0$  and  $p_1$  using balanced homodyne detection.

Since we cannot generate an EPR pair  $e^{\bar{u}_1 \bar{u}_2}$  with an infinite squeezing parameter  $g = \infty$ , the ideal  $q = 1$  case of the squeezed state  $e^{q\bar{u}_1 \bar{u}_2}$ ,  $q = \tanh g < 1$ , we cannot have complete teleportation  $\psi_{\text{out}} = \psi (= \psi_{\text{in}})$ . We have to measure the distance between  $\psi$  and  $\psi_{\text{out}}$ . For density operators  $\rho$  (positive operator with  $\text{tr } \rho = 1$ ) and  $\sigma$ , the fidelity  $F(\rho, \sigma)$ , given by

$$F(\rho, \sigma) = \text{tr } \sqrt{\rho^{1/2} \sigma \rho^{1/2}}$$

is an accepted measure for such a distance. Let  $|\psi\rangle = |\gamma\rangle$  be a coherent state, and

$$|\phi\rangle = \sum_{n=0}^{\infty} q^n D(\alpha) |n\rangle \langle n| D(\alpha)^\dagger |\gamma\rangle.$$

For  $\rho = |\gamma\rangle\langle\gamma|$  and  $\sigma = |\phi\rangle\langle\phi|$ ,

$$\sqrt{\rho^{1/2} \sigma \rho^{1/2}} = \sqrt{\rho \sigma \rho} = \sqrt{|\gamma\rangle\langle\gamma| |\phi\rangle\langle\phi| |\gamma\rangle\langle\gamma|} = |\langle\gamma|\phi\rangle| |\gamma\rangle\langle\gamma|$$

and the fidelity  $F(\rho, \sigma)$  equals  $|\langle\gamma|\phi\rangle|^2$ . The inner product  $\langle\gamma|\phi\rangle$  is easily calculated, using relation (4.1):

$$\begin{aligned} \langle\gamma|\phi\rangle &= \sum_{n=0}^{\infty} q^n \langle\gamma| D(\alpha) |n\rangle \langle n| D(\alpha)^\dagger |\gamma\rangle = \int \int e^{-|\alpha|^2/2} e^{-\bar{\alpha}u} e^{-|\gamma|^2/2} e^{\bar{\gamma}(\bar{u}+\bar{\alpha})} e^{q\bar{u}v} \\ &\quad \times e^{-|\alpha|^2/2} e^{-\alpha\bar{v}} e^{-|\gamma|^2/2} e^{\gamma(\bar{v}+\bar{\alpha})} e^{-\bar{u}u} e^{-\bar{v}v} \frac{d\bar{u}du}{2\pi i} \frac{d\bar{v}dv}{2\pi i} \\ &= \exp(1 - q) \{ \alpha\bar{\gamma} + \bar{\alpha}\gamma - |\alpha|^2 - |\gamma|^2 \}. \end{aligned}$$

## 7. EXPERIMENT, CONTROVERSY AND LOCALITY

**7.1. The Experiment.** Here, we briefly explain the experiment by the Caltech group, Furusawa et al. [10]. The teleported state is not an optical beam itself but a modulation sideband of a bright optical beam generated by an electro-optical modulator, because the frequency of the optical beam is too high ( $\omega/\pi = 300\text{THz}$ , wavelength 1000 nm) to handle directly. So, the optical beam is treated only as a carrier and the quantum states are discussed using sideband frequency. The light from a single-frequency titanium sapphire ( $\text{TiAl}_2\text{O}_3$ ) laser at 860 nm (frequency  $\omega_L$ ) serves as the primary source for all fields in the experiment. 90 percent of the laser output with frequency  $\omega_L$  is directed to a frequency-doubling cavity to generate blue light at  $2\omega_L$ . This output then splits into two beams that serve as harmonic pumps for parametric down-conversion,  $2\omega_L \rightarrow \omega_L \pm \Omega$ , within the optical parametric oscillator (OPO).

- (1) Thus the two independent squeezed beams which are supposed to be represented by  $e^{q\bar{u}_1^2}$  and  $e^{-q\bar{u}_2^2}$  at 860 nm are generated by optical parametric oscillators (amplifiers), where  $q = \tanh g$  and  $g$  is called the squeezing parameter. These beams are sent through a beamsplitter obtaining the EPR beam  $e^{q\bar{u}_1\bar{u}_2}$  and the outcomes are sent to Alice (variable  $\bar{u}_1$ ) and Bob (variable  $\bar{u}_2$ ).

Victor generates a coherent sideband  $|\psi\rangle = e^{-|\alpha|^2/2} e^{\alpha\bar{u}_0}$  at frequency  $(\omega \pm \Omega)/2\pi$  with  $\Omega/2\pi = 2.9\text{ MHz}$  by an electro-optical modulator, and sends it to Alice.

If  $e^{it\omega}$  represents the carrier and  $\alpha e^{\pm it\Omega}$  the modulation, then the total system is represented by

$$\alpha e^{\pm it\Omega} e^{it\omega} = \alpha e^{i(\omega \pm \Omega)t}.$$

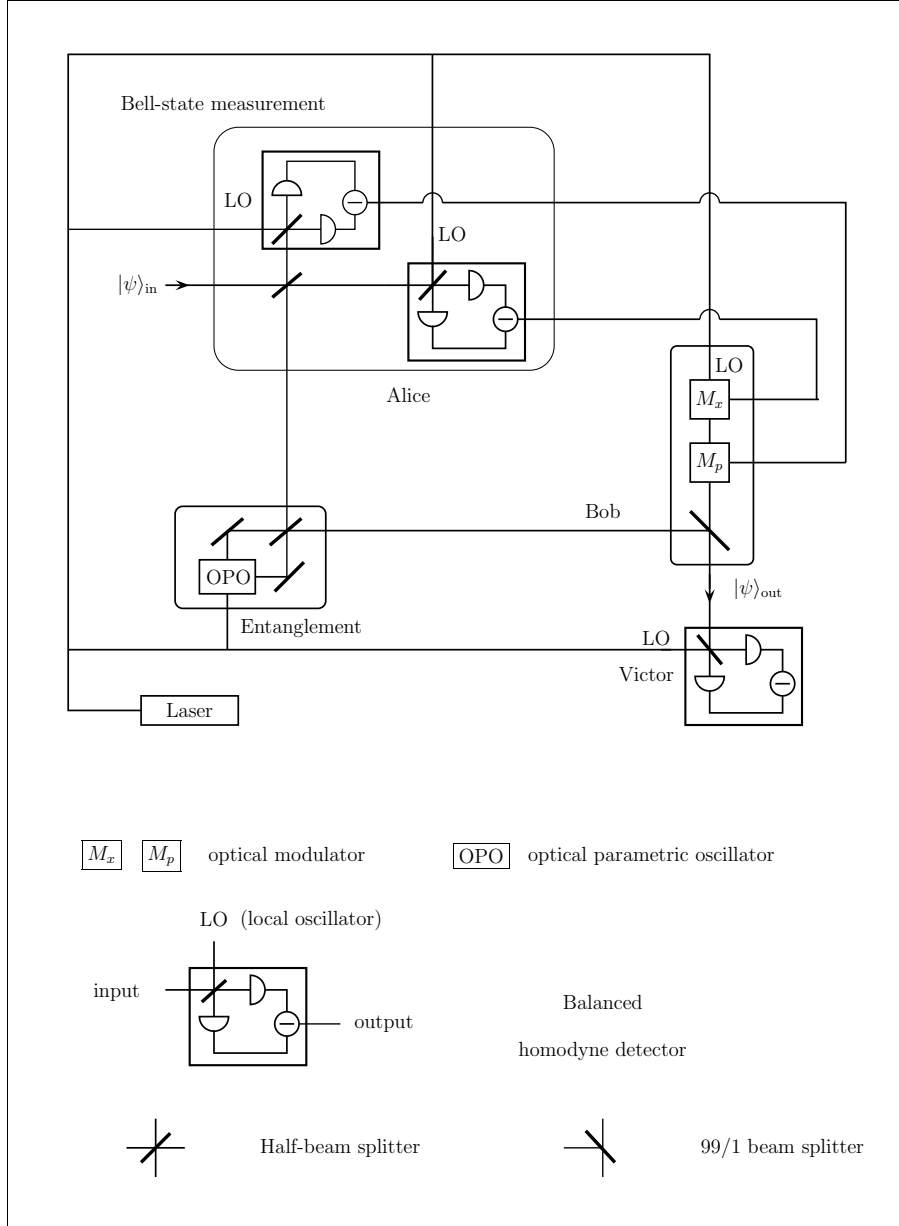


FIGURE 3. Experimental scheme for quantum teleportation performed by Caltech group. The laser field is shared by all parties.

- (2) (a) Alice sends two beams,  $e^{-|\alpha|/2}e^{\alpha\bar{u}_0}$  and her EPR beam, through a half-beamsplitter  $e^{iH_{\text{hbs}}}$  discussed in Section 5, obtaining

$$e^{-|\alpha|/2}e^{\alpha(\bar{u}_0+\bar{u}_1)\sqrt{2}}e^{q(-\bar{u}_0+\bar{u}_1)\bar{u}_2/\sqrt{2}}.$$

- (b) Alice uses two sets of balanced homodyne detectors ( $D_{x_0}, D_{p_1}$ ) discussed in Section 5 to make a joint measurement of the amplitude  $x_0$  and  $p_1$  discussed in Section 6.
- (3) Alice sends the outcomes  $(x_-, p_+)$  of the detectors to Bob.
- (4) After receiving this classical information from Alice, Bob is able to construct the teleported state  $\rho_{\text{out}}$ . Bob generates a sideband beam at frequency  $(\omega \pm \Omega)/2\pi$  by two electro-optical modulators  $M_x$  (amplitude modulator) and  $M_p$  (phase modulator) with suitable complex amplitude  $|\beta\rangle = e^{-|\beta|^2/2} e^{i\beta u_3}$ . Then he sends  $|\beta\rangle$  and his EPR beam through a beamsplitter of refractivity 0.99 obtaining the state  $\rho_{\text{out}}$ .
- (5) Victor detects the state  $\rho_{\text{out}}$  by his own balanced homodyne detector  $D_V$  and compares it with his original state  $|\psi\rangle$ .

The result obtained by Furusawa, A., et al. is that the fidelity  $F(|\psi\rangle, \rho_{\text{out}})$  of the states  $|\psi\rangle$  and  $\rho_{\text{out}}$  is

$$F(|\psi\rangle, \rho_{\text{out}}) = 0.58 \pm 0.02.$$

Bowen, W.P., et al. [5] had the fidelity of  $0.64 \pm 0.02$  and Takei, N., et al. [16] achieved the fidelity of  $0.70 \pm 0.02$ .

**7.2. Controversy.** After the experimental demonstration of continuous variable quantum teleportation (CVQT) was reported [10], there was a controversy over its validity on the ground of intrinsic phase indeterminacy of the laser field [12]. The laser field is often assumed to be a coherent state having a fixed phase, but [12] shows that the steady-state solution of the master equation in the quantum theory of the laser shows that the phase of the laser field inside the cavity is completely unknown and genuine CVQT cannot be achieved using conventional laser sources, due to an absence of optical coherence. Furthermore, the same laser source is used for (i) producing Victor's state for teleportation, (ii) pumping the nonlinear crystal that produces a two-mode squeezed light field serving as the shared EPR state, (iii) supplying local oscillator (LO) fields for both of Alice's homodyne measurements, and (iv) providing Bob with a coherent field to mix with his portion of the EPR beam to reconstruct Victor's state.

The ideal scheme for CVQT [6] is explained in Fig. 2 and an explanation of the experimental simulation of CVQT [10] is provided in Fig. 3.

For the claim [12], there appeared a counterargument [18] which says that the standard description of the laser field used in [12] is insufficient to understand CVQT with a laser. They found that the laser light has the random phase only in the cavity, and outside the cavity, it has similar phase character as the coherent state has, and therefore a conventional laser can be used for CVQT.

Furthermore, [9] says that the laser field outside the cavity is a mixed state whose phase is completely unknown, but CVQT with a laser is

valid only if the unknown phase of the laser field is shared among sender's LOs, the EPR state, and receiver's LO.

**7.3. Locality.** Let  $f_i(x)$  be two functions square integrable functions on Euclidean space, i.e., elements in  $L^2(\mathbb{R}^3)$ , and suppose that the support of the function  $f_i$  is contained in a bounded set  $O_i$  of  $\mathbb{R}^3$ . Then for a system in the state represented by the function  $f_i(x)$ , the observable  $x$  (the position of a particle) is always observed in the set  $O_i$ . In such situation, we often say that the state  $f_i$  is localized in  $O_i$ . The state of two particles at  $x_1$  and  $x_2$  which are separated by a long distance is expressed by the tensor product  $f(x_1, x_2) = f_1(x_1)f_2(x_2)$  of two functions  $f_i$  ( $i = 1, 2$ ) whose supports  $O_i$  ( $i = 1, 2$ ) are separated by a long distance.

Ideal quantum teleportation can be considered as the following process: The information contained in a quantum state  $|\psi\rangle_{\text{in}}$  (which is localized in a region  $O_1$ ) is sent to another region  $O_2$  separated from  $O_1$  by a long distance and produces a state  $|\psi\rangle_{\text{out}}$  (which is localized in  $O_2$ ) and contains all the information of  $|\psi\rangle_{\text{in}}$ .

In the framework of teleportation of qubits in Section 2 where a finite dimensional Hilbert space is used, it seems difficult to define a state localized in some bounded region. In the framework of CVQT in Section 6, though an infinite dimensional Hilbert space is used, we manipulate only monochromatic laser beams. When the frequency of the laser beam is determined, the location of the laser beam is completely unknown. Up to now, one has considered quantum teleportation without the notion of locality. In a precise formulation of quantum teleportation which takes locality into account, a quantum theory of infinitely many degrees of freedom might be needed, e.g., a quantum field theory might be necessary.

It seems that the theory of lasers is not satisfactory in its application to quantum teleportation, and therefore, many controversies appear around CVQT which uses laser beams. In this context to, a theory of quantum teleportation based on quantum field theory might be preferable.

## APPENDIX A.

**A.1. Generalized vectors and generalized eigen-states.** The theory of the generalized vectors is developed in [19]. Let  $\mathcal{S}(\mathbb{R})$  the Schwartz space of fast decaying infinitely often differentiable functions on the real line  $\mathbb{R}$  and  $L^2(\mathbb{R})$  the Hilbert space of square integrable functions on  $\mathbb{R}$ . Then

$$\mathcal{S}(\mathbb{R}) \subset L^2(\mathbb{R}) \subset \mathcal{S}'(\mathbb{R})$$

is a rigged Hilbert space (see [19]). Here  $\mathcal{S}'(\mathbb{R})$  denotes the topological dual of  $\mathcal{S}(\mathbb{R})$ .

The elements  $F$  of  $\mathcal{S}'$  are called generalized vectors. Suppose that an operator  $A$  on  $L^2(\mathbb{R})$  maps  $\mathcal{S}(\mathbb{R})$  into  $\mathcal{S}(\mathbb{R})$ . Then a generalized vector  $F_\lambda \in \mathcal{S}'(\mathbb{R})$  is called a generalized eigen-vector of  $A$  corresponding to an eigen-value  $\lambda$ , if

$$F_\lambda(A\phi) = \lambda F_\lambda(\phi)$$

holds for every  $\phi \in \mathcal{S}(\mathbb{R})$ .

Let

$$E_\lambda = \{F \in \mathcal{S}'(\mathbb{R}); F(A\phi) = \lambda F(\phi)\}$$

be the space of generalized eigen-vectors of  $A$  corresponding to the eigen-value  $\lambda$ . We associate with each element  $\phi \in \mathcal{S}(\mathbb{R})$  and each number  $\lambda$  a linear functional  $\tilde{\phi}_\lambda$  on  $E_\lambda$ , taking the value  $F_\lambda(\phi)$  on the element  $F_\lambda$  of  $E_\lambda$ . We call the correspondence  $\phi \rightarrow \tilde{\phi}_\lambda$  the spectral decomposition of the element  $\phi$  corresponding to the operator  $A$ .

If  $\tilde{\phi}_\lambda \equiv 0$  implies  $\phi = 0$ , then we say that the set of generalized eigen-vectors of the operator  $A$  is complete.

Let  $p = -id/dx$  be the self-adjoint generator of translations in  $L^2(\mathbb{R})$  (more accurately  $p$  is the self-adjoint realization of the differential operator  $-id/dx$  on  $\mathcal{S}(\mathbb{R})$ ). Then the identity

$$(-id/dx)e^{i\lambda x} = \lambda e^{i\lambda x}, \quad \lambda \in \mathbb{R}$$

shows that  $e^{i\lambda x}$  is an eigen-function of the operator  $p$  corresponding to the eigen-value  $\lambda$ . The function  $e^{i\lambda x}$  does not belong to  $L^2(\mathbb{R})$ , but belongs to  $\mathcal{S}'(\mathbb{R}) \ni F_\lambda = e^{i\lambda x}$ . The relation

$$\begin{aligned} F_\lambda(p\phi) &= \int \overline{e^{i\lambda x}}(-id/dx)\phi(x)dx = \int \overline{(-id/dx)e^{i\lambda x}}\phi(x)dx \\ &= \lambda \int \overline{e^{i\lambda x}}\phi(x)dx = \lambda F_\lambda(\phi) \end{aligned}$$

shows that  $F_\lambda = e^{i\lambda x}$  is a generalized eigen- vector of  $p$  corresponding to the eigen-value  $\lambda$ . The spectral decomposition  $\tilde{\phi}_\lambda$  of  $\phi$  is the Fourier transformation of  $\phi$ .

$$\tilde{\phi}_\lambda = F_\lambda(\phi) = \int \overline{e^{i\lambda x}}\phi(x)dx = \int e^{-i\lambda x}\phi(x)dx = \tilde{\phi}(\lambda).$$

The Fourier inversion formula

$$\phi(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\lambda x} \tilde{\phi}(\lambda) d\lambda$$

shows that  $\tilde{\phi}_\lambda \equiv 0$  implies  $\phi = 0$ , that is,  $\{e^{i\lambda x}; \lambda \in \mathbb{R}\}$  is a complete set of generalized eigen-vectors of  $p$ .

Though it is not possible to find the value of the momentum  $p$  (continuous variable) to be precisely  $\lambda$ , it is convenient to say that if the value  $\lambda$  of the momentum  $p$  is found for the state  $|\phi\rangle = \phi(x)$ , the state after the measurement is  $|\lambda\rangle\langle\lambda|\phi\rangle = e^{i\lambda x}\tilde{\phi}(\lambda)/2\pi$  in the same way as the discrete variables, where  $|\lambda\rangle = e^{i\lambda x}/\sqrt{2\pi}$  is a generalized vector.

Actually, we can only say that  $\lambda$  is contained in the interval  $[a, b]$ . In

that case, using the above generalized eigen-states, the state after the measurement is

$$\int_a^b d\lambda |\lambda\rangle \langle \lambda | \phi\rangle / \sqrt{\int_a^b d\lambda |\langle \lambda | \phi\rangle|^2}$$

. Thus we have the so-called projective measurement, whose name comes from the fact that  $P = \int_a^b d\lambda |\lambda\rangle \langle \lambda|$  is a projection. If we think that the edge of  $[a, b]$  is not sharp, then we take  $M = \int_a^b d\lambda \chi(\lambda) |\lambda\rangle \langle \lambda|$  as a measurement operator, where  $\chi(\lambda)$  is a  $C^\infty$ -function with the support contained in  $[a - \epsilon, b + \epsilon]$  and  $\chi(\lambda) = 1$  for  $\lambda \in [a + \epsilon, b - \epsilon]$  and  $\epsilon > 0$ . Then we have the following generalized measurement (see [20]).

The state after the measurement is

$$M|\phi\rangle / \sqrt{\langle \phi | M^* M | \phi \rangle}.$$

If we introduce the operators

$$a = \frac{1}{\sqrt{2}} \left( x + \frac{d}{dx} \right), \quad a^* = \frac{1}{\sqrt{2}} \left( x - \frac{d}{dx} \right),$$

the topology of  $\mathcal{S}(\mathbb{R})$  is defined by the system of norms

$$\|\phi\|_r^2 = \langle \phi | (1 + a^* a)^r | \phi \rangle$$

for any  $r \in \mathbb{N}$  (see Theorem V.13 of [21]).

In our case of a rigged Hilbert space  $\Phi \subset \mathcal{H} \subset \Phi'$ ,  $\Phi$  is the set of vectors of the form

$$\phi = \sum_{n=0}^{\infty} c_n |n\rangle$$

satisfying

$$\|\phi\|_r^2 = \langle \phi | (1 + a^* a)^r | \phi \rangle = \sum_{n=0}^{\infty} (1 + n^2)^r |c_n|^2 < \infty$$

for any  $r \in \mathbb{N}$ .  $\Phi$  is a countably normed space. Since

$$\sum_{n=0}^{\infty} (1 + n^2)^r |\lambda|^n / n! < \infty,$$

$e^{\lambda \bar{u}} \in \Phi$  for any  $\lambda \in \mathbb{C}$ , the complex numbers.

For  $|q| < 1$  we have

$$\left\| \sum_{n=0}^{\infty} q^n |n\rangle_0 \otimes |n\rangle_1 \right\|^2 = \sum_{n=0}^{\infty} q^{2n} = \frac{1}{1 - q^2},$$

but for  $q = 1$  this series is divergent:

$$\left\| \sum_{n=0}^{\infty} |n\rangle_0 \otimes |n\rangle_1 \right\|^2 = \sum_{n=0}^{\infty} 1 = \infty.$$

In this sense the vector

$$\pi^{-1/2} e^{\bar{v}_0 \bar{v}_1} = \pi^{-1/2} \sum_{n=0}^{\infty} \frac{(\bar{v}_0 \bar{v}_1)^n}{n!} = \pi^{-1/2} \sum_{n=0}^{\infty} |n\rangle_0 \otimes |n\rangle_1$$

is a generalized vector. The unitary operator  $e^{iH_{\text{hbs}}}$  sends the generalized vector

$$\sum_{n=0}^{\infty} (D(-\alpha)|n\rangle_0) \otimes |n\rangle_1 = e^{-|\alpha|^2/2} e^{\bar{v}_0 \bar{v}_1} e^{-\bar{\alpha} \bar{v}_1} e^{\alpha \bar{v}_0}$$

to the generalized vector

$$= e^{-(x_-^2 + p_+^2)/2} e^{-\bar{u}_0^2/2} e^{\bar{u}_1^2/2} e^{\sqrt{2}x_- \bar{u}_0} e^{i\sqrt{2}p_+ \bar{u}_1} = \pi^{1/2} |x_- \rangle \otimes |p_+ \rangle,$$

where  $\alpha = x_- + ip_+$  and

$$|x_- \rangle = \pi^{-1/4} e^{-x_-^2/2} e^{-\bar{u}_0^2/2} e^{\sqrt{2}x_- \bar{u}_0}$$

is the generalized eigen-state of the operator

$$x_0 = \frac{1}{\sqrt{2}} \left( \frac{\partial}{\partial \bar{u}_0} + \bar{u}_0 \right)$$

with eigen-value  $x_-$ , and

$$|p_+ \rangle = \pi^{-1/4} e^{-p_+^2/2} e^{\bar{u}_1^2/2} e^{i\sqrt{2}p_+ \bar{u}_1}$$

is the generalized eigen-state of the operator

$$p_1 = \frac{1}{\sqrt{2}i} \left( \frac{\partial}{\partial \bar{u}_1} - \bar{u}_1 \right)$$

with eigen-value  $p_+$ . In the usual Schrödinger representation,  $|p_+ \rangle$  is represented by  $(2\pi)^{-1/2} e^{ip_+ x}$  and is a generalized eigen-vector of  $p = -id/dx$  as explained above. Since  $\mathcal{H}$  is a separable Hilbert space,  $\mathcal{H}$  has a countable basis. But for the completeness,  $\mathcal{H}$  must have uncountably many orthogonal generalized vectors.

## A.2. Proofs of theorems 1 - 4, Section 4.

A.2.1. *Proof of Theorem 1.* First, consider the transformation

$$b_j = a_j + f_j, \quad b_j^\dagger = a_j^\dagger + \bar{f}_j$$

for complex numbers  $f_j$ . We want to have a unitary operator  $U$  such that

$$b_j U = U a_j, \quad b_j^\dagger U = U a_j^\dagger \iff b_j = U a_j U^\dagger, \quad b_j^\dagger = U a_j^\dagger U^\dagger.$$

We assume that the operator  $U$  is defined by a kernel  $U(\bar{u}, v)$ , that is,

$$(Ug)(\bar{u}) = \int U(\bar{u}, v) g(v) \prod_{j=1}^n e^{-\bar{v}_j v_j} \frac{d\bar{v}_j dv_j}{2\pi i}.$$



Then we have, by integration by part,

$$\begin{aligned} & \int U(\bar{u}, v) \left\{ \frac{\partial}{\partial \bar{v}_j} g(\bar{v}) \right\} \prod_{j=1}^n e^{-\bar{v}_j v_j} \frac{d\bar{v}_j dv_j}{2\pi i} = \\ & - \int g(\bar{v}) \frac{\partial}{\partial \bar{v}_j} \left\{ U(\bar{u}, v) \prod_{j=1}^n e^{-\bar{v}_j v_j} \right\} \frac{d\bar{v}_j dv_j}{2\pi i} = \int v_j U(\bar{u}, v) g(\bar{v}) \prod_{j=1}^n e^{-\bar{v}_j v_j} \frac{d\bar{v}_j dv_j}{2\pi i}, \end{aligned}$$

and similarly

$$\begin{aligned} & \int U(\bar{u}, v) \bar{v}_j g(\bar{v}) \prod_{j=1}^n e^{-\bar{v}_j v_j} \frac{d\bar{v}_j dv_j}{2\pi i} \\ & = - \int U(\bar{u}, v) \left\{ \frac{\partial}{\partial v_j} g(\bar{v}) \prod_{j=1}^n e^{-\bar{v}_j v_j} \right\} \frac{d\bar{v}_j dv_j}{2\pi i} \\ & = \int \left\{ \frac{\partial}{\partial v_j} U(\bar{u}, v) \right\} g(\bar{v}) \prod_{j=1}^n e^{-\bar{v}_j v_j} \frac{d\bar{v}_j dv_j}{2\pi i}. \end{aligned}$$

Thus, it is clear that one should have the following correspondence:

$$a_j U \leftrightarrow \frac{\partial}{\partial \bar{u}_j} U(\bar{u}, v), \quad a_j^\dagger U \leftrightarrow \bar{u}_j U(\bar{u}, v),$$

$$U a_j \leftrightarrow v_j U(\bar{u}, v), \quad U a_j^\dagger \leftrightarrow \frac{\partial}{\partial v_j} U(\bar{u}, v).$$

If we assume that the kernel of the operator  $U$  has the form

$$U(\bar{u}, v) = c \exp \sum_{j=1}^n (\bar{u}_j v_j + v_j \phi_j + \bar{u}_j \psi_j)$$

for complex numbers  $\phi_j$  and  $\psi_j$ , then we get indeed

$$b_j U(\bar{u}, v) = \left( \frac{\partial}{\partial \bar{u}_j} + f_j \right) U(\bar{u}, v) = v_j U(\bar{u}, v),$$

$$b_j^* U(\bar{u}, v) = (\bar{u}_j + \bar{f}_j) U(\bar{u}, v) = \frac{\partial}{\partial v_j} U(\bar{u}, v).$$

Since

$$\frac{\partial}{\partial \bar{u}_j} U(\bar{u}, v) = (v_j + \psi_j) U(\bar{u}, v),$$

$$\frac{\partial}{\partial v_j} U(\bar{u}, v) = (\bar{u}_j + \phi_j) U(\bar{u}, v)$$

equating the coefficients for  $\bar{u}_j$  and  $v_j$  we find  $\psi_j = -f_j$  and  $\phi_j = \bar{f}_j$ .

Thus the result is

$$U(\bar{u}, v) = c \exp \sum_{j=1}^n (\bar{u}_j v_j + v_j \bar{f}_j - \bar{u}_j f_j).$$

Applying  $U$  to the vacuum  $f(\bar{v}) = 1$  produces

$$\begin{aligned}
F(\bar{u}) &= U|0\rangle = \int c \exp \sum_{j=1}^n (\bar{u}_j v_j + v_j \bar{f}_j - \bar{u}_j f_j) \prod_{j=1}^n e^{-\bar{v}_j v_j} \frac{d\bar{v}_j dv_j}{2\pi i} \\
&= c \exp \left\{ - \sum_{j=1}^n \bar{u}_j f_j \right\} \int \exp \left\{ \sum_{j=1}^n (\bar{u}_j + \bar{f}_j) v_j \right\} \prod_{j=1}^n e^{-\bar{v}_j v_j} \frac{d\bar{v}_j dv_j}{2\pi i} \\
&= c \exp \left\{ - \sum_{j=1}^n \bar{u}_j f_j \right\}. \\
\|F\|^2 &= |c|^2 \int \exp \left\{ - \sum_{j=1}^n [u_j \bar{f}_j + \bar{u}_j f_j] \right\} \prod_{j=1}^n e^{-\bar{u}_j u_j} \frac{d\bar{u}_j du_j}{2\pi i} \\
&= |c|^2 \int \exp - \left\{ \sum_{j=1}^n [(\bar{u}_j + \bar{f}_j)(u_j + f_j) - \bar{f}_j f_j] \right\} \frac{d\bar{u}_j du_j}{2\pi i} \\
&= |c|^2 \exp \left\{ \sum_{j=1}^n \bar{f}_j f_j \right\}.
\end{aligned}$$

The normalization of  $F$ , i.e.,  $\|F\| = 1$ , requires

$$c = \theta \exp \left\{ (-1/2) \sum_{j=1}^n \bar{f}_j f_j \right\}, \quad |\theta| = 1,$$

and we have the following theorem.

*A.2.2. Proof of Theorem 2.* Now we consider more general linear canonical transformation (Bogoliubov transformation)

$$b_j = \sum_{k=1}^n (\Phi_{jk} a_k + \Psi_{jk} a_k^*), \quad b_j^* = \sum_{k=1}^n (\bar{\Phi}_{jk} a_k^* + \bar{\Psi}_{jk} a_k)$$

satisfying

$$[b_j, b_k] = [b_j, b_k] = 0, \quad [b_j, b_k^*] = \delta_{jk},$$

and find the kernel  $U(\bar{u}, v)$  of the unitary operator  $U$  which implements the linear canonical transformation

$$b_j U = U a_j, \quad b_j^* U = U a_j^*.$$

From the above commutation relations, we have

$$\begin{aligned}
0 &= [b_j, b_k] = \sum_{i=1, m=1}^n [\Phi_{ji} a_i + \Psi_{ji} a_i^*, \Phi_{km} a_m + \Psi_{km} a_m^*] \\
&= \sum_{l=1, m=1}^n (\Phi_{jl} \Psi_{km} [a_l, a_m^*] + \Psi_{jl} \Phi_{km} [a_l^*, a_m])
\end{aligned}$$

$$\begin{aligned}
&= \sum_{l=1, m=1}^n (\Phi_{jl} \Psi_{km} \delta_{lm} - \Psi_{jl} \Phi_{km} \delta_{lm}) = \sum_{m=1}^n (\Phi_{jm} \Psi_{km} - \Psi_{jm} \Phi_{km}) \\
&\delta_{jk} = [b_j, b_k^*] = \sum_{i=1, m=1}^n [\Phi_{jl} a_l + \Psi_{jl} a_l^*, \bar{\Phi}_{km} a_m^* + \bar{\Psi}_{km} a_m] \\
&= \sum_{i=1, m=1}^n (\Phi_{jl} \bar{\Phi}_{km} [a_l, a_m^*] + \Psi_{jl} \bar{\Psi}_{km} [a_l^*, a_m]) \\
&= \sum_{i=1, m=1}^n (\Phi_{jl} \bar{\Phi}_{km} \delta_{lm} - \Psi_{jl} \bar{\Psi}_{km} \delta_{lm}) = \sum_{m=1}^n (\Phi_{jm} \bar{\Phi}_{km} - \Psi_{jm} \bar{\Psi}_{km}).
\end{aligned}$$

These relations can be described by the matrix notation:

$$(A.1) \quad 0 = \Phi \Psi^T - \Psi \Phi^T, \quad I = \Phi \Phi^* - \Psi \Psi^*.$$

We want to have a unitary operator  $U$  such that

$$b_j U = U a_j, \quad b_j^* U = U a_j^*.$$

We assume that the kernel of the operator  $U$  has the form

$$U(\bar{u}, v) = c \exp \left( \frac{1}{2} (v \ \bar{u}) \begin{pmatrix} A^{11} & A^{12} \\ A^{21} & A^{22} \end{pmatrix} \begin{pmatrix} v \\ \bar{u} \end{pmatrix} \right), \quad A^{jk} = {}^t A^{kj}.$$

Then we have

$$b_j U(\bar{u}, v) = \sum_{k=1}^n (\Phi_{jk} \frac{\partial}{\partial \bar{u}_k} + \Psi_{jk} \bar{u}_k) U(\bar{u}, v) = v_j U(\bar{u}, v).$$

Since

$$\frac{\partial}{\partial \bar{u}_k} U(\bar{u}, v) = \sum_{m=1}^n (A_{km}^{21} v_m + A_{km}^{22} \bar{u}) U(\bar{u}, v),$$

equating the coefficients for  $\bar{u}_l$  and  $v_l$  we have

$$\Phi A^{22} + \Psi = 0, \quad \Phi A^{21} = I.$$

In the same way, from the equation

$$b_j^* U(\bar{u}, v) = \sum_{k=1}^n (\bar{\Phi}_{jk} a_k^* + \bar{\Psi}_{jk} a_k) U(\bar{u}, v) = \frac{\partial}{\partial v_j} U(\bar{u}, v),$$

we deduce

$$\bar{\Psi} A^{22} + \bar{\Phi} = A^{12}, \quad \bar{\Psi} A^{21} = A^{11}.$$

This gives

$$A^{22} = -\Phi^{-1} \Psi, \quad A^{21} = \Phi^{-1}, \quad A^{11} = \bar{\Psi} \Phi^{-1}.$$

Applying  $U$  to the vacuum  $|0\rangle$ , we have

$$F(\bar{u}) = U|0\rangle = c \int \exp \left( \frac{1}{2} (v \ \bar{u}) \begin{pmatrix} A^{11} & A^{12} \\ A^{21} & A^{22} \end{pmatrix} \begin{pmatrix} v \\ \bar{u} \end{pmatrix} \right) \prod_{j=1}^n e^{-\bar{v}_j v_j} \frac{d\bar{v}_j dv_j}{2\pi i}$$

$$= c \exp(1/2) \{-\bar{u}\Phi^{-1}\Psi\bar{u}\} \int \exp(1/2) \{2\bar{u}\Phi^{-1}v + v\bar{\Psi}\Phi^{-1}v - \bar{v}v\} \prod_{j=1}^n \frac{d\bar{v}_j dv_j}{2\pi i}.$$

The integral is

$$\begin{aligned} & \int \exp(-1/2) \left\{ (v \ \bar{v}) \begin{pmatrix} -\bar{\Psi}\Phi^{-1} & I \\ I & 0 \end{pmatrix} \begin{pmatrix} v \\ \bar{v} \end{pmatrix} + 2\bar{u}\Phi^{-1}v \right\} \prod_{j=1}^n \frac{d\bar{v}_j dv_j}{2\pi i} \\ &= \left[ \det \begin{pmatrix} I & \bar{\Psi}\Phi^{-1} \\ 0 & I \end{pmatrix} \right]^{-1/2} \\ & \times \exp \left[ (1/2)(0 \ 2\bar{u}\Phi^{-1}) \begin{pmatrix} 0 & I \\ I & \bar{\Psi}\Phi^{-1} \end{pmatrix} \begin{pmatrix} 0 \\ 2\bar{u}\Phi^{-1} \end{pmatrix} \right] = 1, \end{aligned}$$

where we used the following formula of Gaussian integral:

$$\begin{aligned} & \int \exp \left( \frac{-1}{2} (v \ \bar{v}) \begin{pmatrix} A^{11} & A^{12} \\ A^{21} & A^{22} \end{pmatrix} \begin{pmatrix} v \\ \bar{v} \end{pmatrix} + (f_1 \ f_2) \begin{pmatrix} v \\ \bar{v} \end{pmatrix} \right) \prod_{j=1}^n \frac{d\bar{v}_j dv_j}{2\pi i} \\ &= \left[ \det \begin{pmatrix} A^{21} & A^{22} \\ A^{11} & A^{12} \end{pmatrix} \right]^{-1/2} \exp(1/2) \left\{ (f_1 \ f_2) \begin{pmatrix} A^{11} & A^{12} \\ A^{21} & A^{22} \end{pmatrix}^{-1} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \right\}. \end{aligned}$$

Now we calculate

$$\begin{aligned} \|F\|^2 &= |c|^2 \int \exp -\{(1/2)(u\bar{\Phi}^{-1}\bar{\Psi}u + \bar{u}\Phi^{-1}\Psi\bar{u}) + \bar{u}u\} \prod_{j=1}^n \frac{d\bar{u}_j du_j}{2\pi i} \\ &= |c|^2 \int \exp \left\{ \frac{-1}{2} (u \ \bar{u}) \begin{pmatrix} \bar{\Phi}^{-1}\bar{\Psi} & I \\ I & \Phi^{-1}\Psi \end{pmatrix} \begin{pmatrix} u \\ \bar{u} \end{pmatrix} \right\} \prod_{j=1}^n \frac{d\bar{u}_j du_j}{2\pi i} \\ &= |c|^2 \left[ \det \begin{pmatrix} I & \Phi^{-1}\Psi \\ \bar{\Phi}^{-1}\bar{\Psi} & I \end{pmatrix} \right]^{-1/2}. \\ \det \begin{pmatrix} I & \Phi^{-1}\Psi \\ \bar{\Phi}^{-1}\bar{\Psi} & I \end{pmatrix} &= \det \left[ \begin{pmatrix} I & 0 \\ -\bar{\Phi}^{-1}\bar{\Psi} & I \end{pmatrix} \begin{pmatrix} I & \Phi^{-1}\Psi \\ \bar{\Phi}^{-1}\bar{\Psi} & I \end{pmatrix} \right] \\ &= \det \left[ \begin{pmatrix} I & -\Phi^{-1}\Psi \\ 0 & I \end{pmatrix} \begin{pmatrix} I & \Phi^{-1}\Psi \\ \bar{\Phi}^{-1}\bar{\Psi} & I \end{pmatrix} \right] = \det \begin{pmatrix} I & -\Phi^{-1}\Psi\bar{\Phi}^{-1}\bar{\Psi} \\ 0 & I \end{pmatrix} \\ &= \det(I - \Phi^{-1}\Psi\bar{\Phi}^{-1}\bar{\Psi}) = \det(I - \Phi^{-1}\Psi\Psi^*\Phi^{*-1}) \\ &= \det(I - \Phi^{-1}(\Phi\Phi^* - I)\Phi^{*-1}) = \det(\Phi^*\Phi)^{-1}, \end{aligned}$$

where we have the relation (A.1). The constant  $c$  is calculated to be

$$c = \theta(\det \Phi\Phi^*)^{-1/4}, \quad |\theta| = 1,$$

and we get the following result.

A.2.3. *Proof of Theorem 3.* Proof. Consider the operator

$$(A.2) \quad a_j(t) = e^{itH} a_j e^{-itH}, \quad a_j^*(t) = e^{itH} a_j^* e^{-itH}.$$

Differentiating with respect to  $t$ , we have

$$\begin{aligned} \frac{1}{i} \frac{da_j(t)}{dt} &= [H, a_j(t)] = [H(t), a_j(t)] = e^{itH} [H, a_j] e^{-itH} \\ &= e^{itH} \{-f_j\} e^{-itH} = -f_j, \\ \frac{1}{i} \frac{da_j^*(t)}{dt} &= [H, a_j^*(t)] = [H(t), a_j^*(t)] = e^{itH} [H, a_j^*] e^{-itH} \\ &= e^{itH} \bar{f}_j e^{-itH} = \bar{f}_j. \end{aligned}$$

Integrating this system, we have

$$a_j(t) = a_j - itf_j, \quad a_j^*(t) = a_j^* + it\bar{f}_j.$$

Thus (A.2) is a linear canonical transformation of Theorem 4.1. Therefore the kernel  $U(\bar{u}, v)$  has the form (4.4). In order to find  $c$  precisely, we differentiate  $U(\bar{u}, v)$  and  $e^{itH}$  and compare them.

$$\begin{aligned} \frac{1}{i} \frac{d}{dt} U(\bar{u}, v) &= \left( \frac{1}{i} \frac{dc}{dt} + c \sum_{j=1}^n (f_j \bar{u}_j + \bar{f}_j v_j) \right) U(\bar{u}, v) \\ &= c \sum_{j=1}^n \left( f_j \bar{u}_j + \bar{f}_j \frac{\partial}{\partial \bar{u}_j} \right) U(\bar{u}, v) = c \left( \sum_{j=1}^n (f_j \bar{u}_j + \bar{f}_j v_j) + it \bar{f}_j f_j \right) U(\bar{u}, v) \end{aligned}$$

As a result we obtain

$$\frac{1}{i} \frac{dc}{dt} = itc \left( \sum_{j=1}^n \bar{f}_j f_j \right), \quad \text{and} \quad c = \exp \left\{ -\frac{1}{2} t^2 \sum_{j=1}^n \bar{f}_j f_j \right\}.$$

A.2.4. *Proof of Theorem 4.* Proof. Consider the operator

$$(A.3) \quad a_j(t) = e^{itH} a_j e^{-itH}, \quad a_j^*(t) = e^{itH} a_j^* e^{-itH}.$$

Differentiating with respect to  $t$ , we have

$$\begin{aligned} \frac{1}{i} \frac{da_j(t)}{dt} &= [H, a_j(t)] = [H(t), a_j(t)] = e^{itH} [H, a_j] e^{-itH} \\ &= e^{itH} \sum_{k=1}^n -\{C_{jk} a_k + B_{jk} a_k^*\} e^{-itH} = - \sum_{k=1}^n \{C_{jk} a_k(t) + B_{jk} a_k^*(t)\}, \\ \frac{1}{i} \frac{da_j^*(t)}{dt} &= [H, a_j^*(t)] = [H(t), a_j^*(t)] = e^{itH} [H, a_j^*] e^{-itH} \\ &= e^{itH} \sum_{k=1}^n \{\bar{B}_{jk} a_k + \bar{C}_{jk} a_k^*\} e^{-itH} = \sum_{k=1}^n \{\bar{B}_{jk} a_k(t) + \bar{C}_{jk} a_k^*(t)\}. \end{aligned}$$

Integrating this system, we have

$$a_j(t) = \sum_{k=1}^n (\Phi_{jk} a_k + \Psi_{jk} a_k^*), \quad a_j^*(t) = \sum_{k=1}^n (\bar{\Phi}_{jk} a_k^* + \bar{\Psi}_{jk} a_k),$$

where

$$\begin{pmatrix} \Phi & \Psi \\ \bar{\Psi} & \bar{\Phi} \end{pmatrix} = \exp \left\{ it \begin{pmatrix} -C & -B \\ \bar{B} & \bar{C} \end{pmatrix} \right\}.$$

Thus, Eq. (A.3) is a linear canonical transformation. In order to find  $c$  precisely, we differentiate  $U(\bar{u}, v)$  and  $e^{itH}$  and compare the results.

$$\begin{aligned} \frac{1}{i} \frac{d}{dt} U(\bar{u}, v) &= \left( \frac{1}{i} \frac{dc}{dt} + c \frac{1}{2i} (v\bar{u}) \frac{d}{dt} \begin{pmatrix} A^{11} & A^{12} \\ A^{21} & A^{22} \end{pmatrix} \begin{pmatrix} v \\ \bar{u} \end{pmatrix} \right) U(\bar{u}, v) \\ &= c \frac{1}{2} \left( \bar{u} B \bar{u} + \frac{\partial}{\partial \bar{u}} \bar{B} \frac{\partial}{\partial \bar{u}} + 2 \bar{u} C \frac{\partial}{\partial \bar{u}} \right) U(\bar{u}, v). \end{aligned}$$

Put  $v = \bar{u} = 0$ . Then we have

$$\frac{1}{i} \frac{dc}{dt} = c \frac{1}{2} \sum_{ij} \bar{B}_{ij} A_{ij}^{22} = c \frac{1}{2} \sum_{ij} A_{ji}^{22} \bar{B}_{ij} = c \frac{1}{2} \text{Tr} A^{22} B = -c \frac{1}{2} \text{Tr} (\Phi^{-1} \Psi B).$$

We shall seek  $c$  in the form  $c = (\det M)^{-1/2}$ . From the formula  $\det M = e^{\text{Tr} \log M}$ , we find

$$\begin{aligned} \frac{d}{dt} \det M &= \left( \frac{d}{dt} \text{Tr} \log M \right) \det M = \text{Tr} \left( \frac{d}{dt} \log M \right) \det M = \\ &= \text{Tr} \left( M^{-1} \frac{dM}{dt} \right) \det M. \end{aligned}$$

Thus we obtain the equation for  $M$ :

$$\frac{1}{i} \text{Tr} \left( M^{-1} \frac{dM}{dt} \right) = \text{Tr} (\Phi^{-1} \Psi B).$$

Recalling that the operators  $\Phi, \Psi$  are the solution of the equation

$$\frac{1}{i} \frac{d}{dt} \begin{pmatrix} \Phi & \Psi \\ \bar{\Psi} & \bar{\Phi} \end{pmatrix} = \begin{pmatrix} \Phi & \Psi \\ \bar{\Psi} & \bar{\Phi} \end{pmatrix} \begin{pmatrix} -C & -B \\ \bar{B} & \bar{C} \end{pmatrix},$$

we have

$$\frac{1}{i} \Phi^{-1} \frac{d\Phi}{dt} = -C + \Phi^{-1} \Psi \bar{B}.$$

We now set  $M = \Phi e^{itC}$ , then we have

$$\begin{aligned} \frac{1}{i} \text{Tr} \left( M^{-1} \frac{dM}{dt} \right) &= \text{Tr} \left( e^{-itC} \Phi^{-1} \left( \frac{1}{i} \frac{d\Phi}{dt} e^{itC} + \Phi C e^{itC} \right) \right) \\ &= \text{Tr} \left( e^{-itC} \Phi^{-1} \frac{1}{i} \frac{d\Phi}{dt} e^{itC} + e^{-itC} C e^{itC} \right) \\ &= \text{Tr} (e^{-itC} (-C + \Phi^{-1} \Psi \bar{B}) e^{itC} + e^{-itC} C e^{itC}) = \text{Tr} (\Phi^{-1} \Psi \bar{B}). \end{aligned}$$

Thus  $c = (\det \Phi e^{itC})^{-1/2}$ . Therefore the kernel  $U(\bar{u}, v)$  has the form (4.5), (4.6).

### A.3. Proofs for Section 6.

A.3.1. *Generalized Bell basis.* First we show completeness:

$$\begin{aligned}
& \pi^{-1} \int e^{-\bar{\alpha}\alpha} e^{\bar{u}_0\bar{u}_1} e^{-\bar{\alpha}\bar{u}_1} e^{\alpha\bar{u}_0} e^{v_0v_1} e^{-\alpha v_1} e^{\bar{\alpha}v_0} dx_- dp_+ \\
&= \pi^{-1} \int e^{-\bar{\alpha}\alpha} e^{\bar{u}_0\bar{u}_1} e^{\bar{\alpha}(v_0-\bar{u}_1)} e^{\alpha(\bar{u}_0-v_1)} e^{v_0v_1} dx_- dp_+ \\
&= \int e^{-\bar{\alpha}\alpha} e^{\bar{u}_0\bar{u}_1} e^{\bar{\alpha}(v_0-\bar{u}_1)} e^{\alpha(\bar{u}_0-v_1)} e^{v_0v_1} \frac{d\bar{\alpha}d\alpha}{2\pi i} \\
&= e^{\bar{u}_0\bar{u}_1} e^{(\bar{u}_0-v_1)(v_0-\bar{u}_1)} e^{v_0v_1} = e^{\bar{u}_0v_0} e^{\bar{u}_1v_1}.
\end{aligned}$$

This is the kernel of the identity operator (see (4.1)). Thus (6.1) is a complete system.

In order to show that (6.1) is an orthogonal system, we rewrite the vector  $\sum_{n=0}^{\infty} (D(\alpha)|n\rangle_0) \otimes |n\rangle_1$  of (6.1) as

$$\begin{aligned}
e^{-|\alpha|^2/2} e^{\bar{v}_0\bar{v}_1} e^{-\bar{\alpha}\bar{v}_1} e^{\alpha\bar{v}_0} &= e^{-|\alpha|^2/2} e^{\bar{v}_0\bar{v}_1} e^{-(x_- - ip_+)\bar{v}_1} e^{(x_- + ip_+)\bar{v}_0} \\
&= e^{-|\alpha|^2/2} e^{\bar{v}_0\bar{v}_1} e^{x_-(\bar{v}_0-\bar{v}_1)} e^{ip_+(\bar{v}_0+\bar{v}_1)} = \\
&= e^{-|\alpha|^2/2} e^{-(\bar{v}_0-\bar{v}_1)^2/4} e^{(\bar{v}_0+\bar{v}_1)^2/4} e^{x_-(\bar{v}_0-\bar{v}_1)} e^{ip_+(\bar{v}_0+\bar{v}_1)}.
\end{aligned}$$

A.3.2. *Proof of Orthogonality relation (6.2).* This is a straightforward calculation:

$$\begin{aligned}
\langle x'_- | x_- \rangle &= \pi^{-1/2} e^{-x'^2_-/2} e^{-x_-^2/2} \int e^{-u_0^2/2} e^{\sqrt{2}x'_- u_0} e^{-\bar{u}_0^2/2} e^{\sqrt{2}x_- \bar{u}_0} e^{-\bar{u}_0 u_0} \frac{d\bar{u}_0 du_0}{2\pi i} \\
&= \pi^{-1/2} e^{-x'^2_-/2} e^{-x_-^2/2} \int e^{-2q^2} e^{\sqrt{2}q(x'_- + x_-)} e^{i\sqrt{2}p(x'_- - x_-)} \frac{dq dp}{\pi} \\
&= \pi^{-1/2} e^{-x'^2_-/2} e^{-x_-^2/2} \int e^{-(q^2 - q(x'_- + x_-))} e^{ip(x'_- - x_-)} \frac{dq dp}{2\pi} \\
&= \pi^{-1/2} e^{-x'^2_-/2} e^{-x_-^2/2} \sqrt{\pi} e^{(x'_- + x_-)^2/4} \delta(x'_- - x_-) = \delta(x'_- - x_-),
\end{aligned}$$

and in a similar way

$$\langle p'_+ | p_+ \rangle = \delta(p'_+ - p_+).$$

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